Look For and Make Use of Structure, Symmetry, and Similarity

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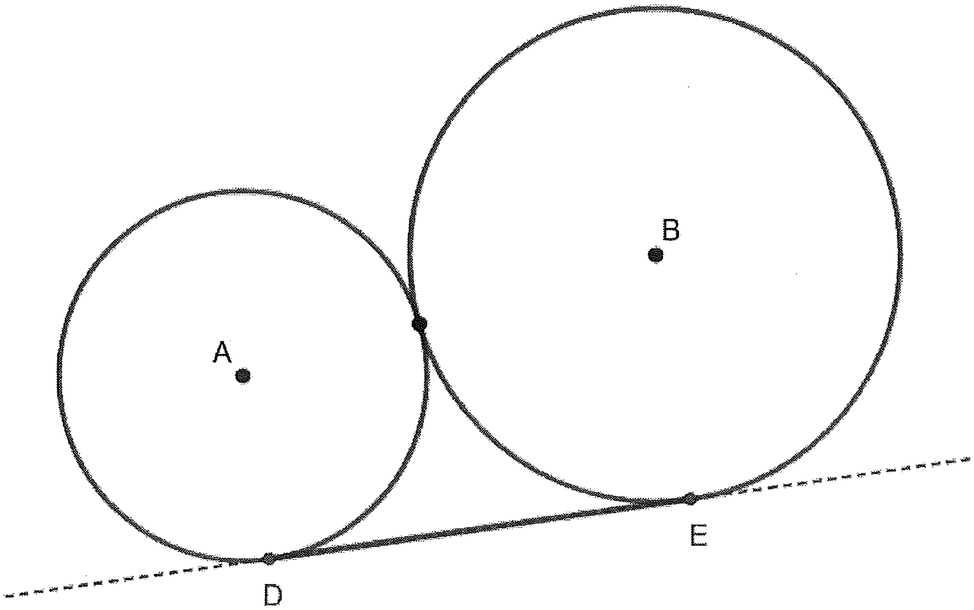
http://faculty.wiu.edu/JR-Olsen/wiu/ - handouts available here electronically.

Common External Tangent

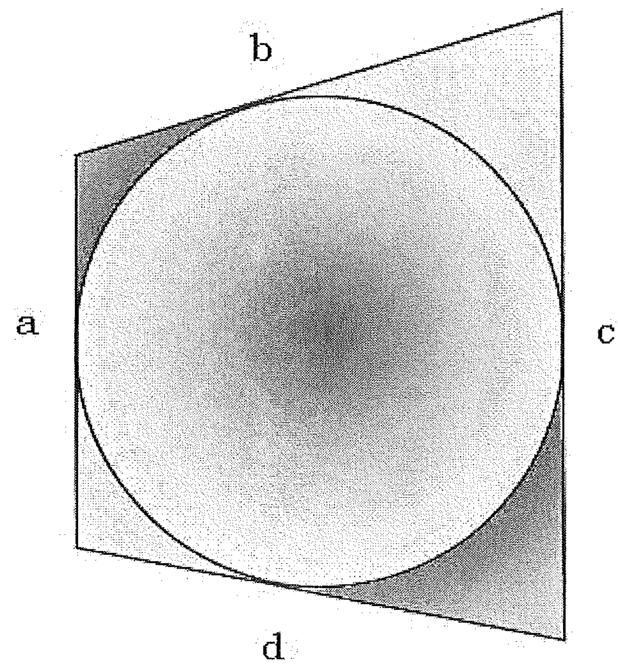
The radius of circle A is 5. The radius of circle B is 7. $\overleftrightarrow{DE}$ is a common external tangent. Find distance DE.



Circumscribe a Quadrilateral

Show the following is true:

If a quadrilateral circumscribes a circle, then the sums of its opposite sides are equal. That is, in the diagram, $a + c = b + d$.



By Cliff Pickover (@pickover)

Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1 Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions,

communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content
The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Jim Olsen, Western Illinois University

A Few Tips for *Looking for and Making Use of Structure*

a) Draw in segments – often to form triangles.

b) Look for symmetry (draw in segments as needed).
- Look for right angles (perpendicular lines).
- Right triangles are nice. Equilateral, 30-60-90°, isosceles, and 45-45-90° triangles are especially nice!
- Look for parallel lines.

c) Draw in radii of circles (*they are all congruent*).

d) If you have a tangent line, a radius drawn to the point of tangency is often very useful (⊥).

e) Pay attention to inscribed angles. They have a measure half the intercepted arc.

f) Look for similar triangles and set up proportions.

g) Be flexible. If something doesn't work, try something different.

Other Resources of Great Problems/Questions

- http://geometry.drjimo.net/ Structure and Symmetry website (by Jim Olsen)
- https://artofproblemsolving.com/wiki/index.php?title=Main_Page Art of Problem Solving Wiki
  - https://artofproblemsolving.com/wiki/index.php/AMC_Problems_and_Solutions AMC Problems and Solutions (AMC8; AMC10, AMC12)
- Twitter:
  - @Cshearer41
  - @pickover
  - @MathCeyhun
  - @ilarrosac
  - @panlepan
  - @Simon_Gregg
  - @puzzlist
  - @jamestanton
  - @StrummingMom
  - @JhuriaMikki
  - @DrOlsen314 (Jim Olsen)
- Sangaku - *Sacred Mathematics: Japanese Temple Geometry*

Handout for ISMAA 2019– SIU-C
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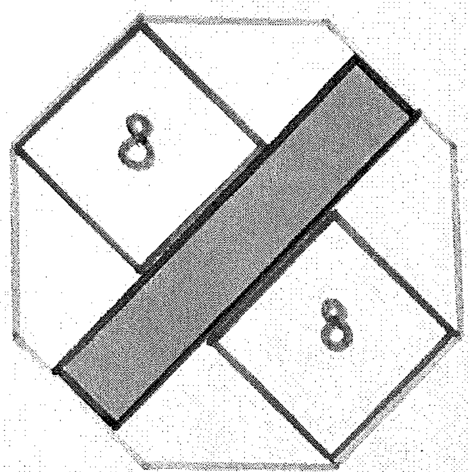
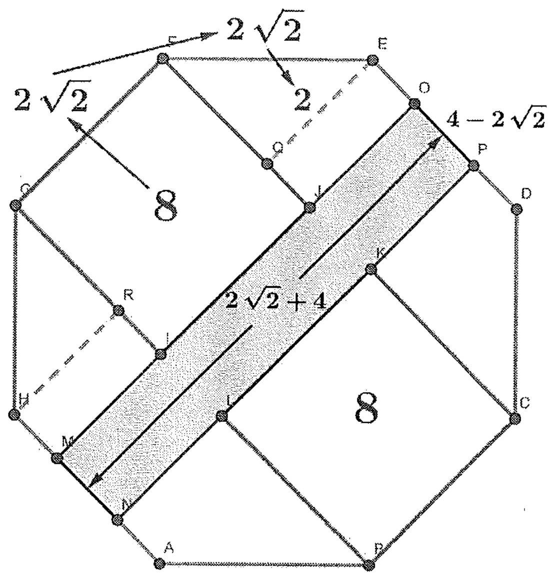
The In-between rectangle in an octagon.

Inside this regular octagon sit two squares of area 8. What's the area of the shaded rectangle?



Source: https://twitter.com/Cshearer41/status/1089227396954419200

Answer: 8
https://twitter.com/sergiosanz001/status/1089249605349507074



Shaded Area= $\left(2\sqrt{2}+4\right)\left(4-2\sqrt{2}\right)=8$

Why These Wonderful Geometry Problems and Audience

By Jim Olsen

Mission Statement:

(My mission is to) Provide a large collection of wonderful geometry problems so that people can develop a connected understanding of mathematics and higher order thinking, see the beauty of mathematics, and experience the joy of mathematical reasoning, as a human experience.

Why – The reasons for these problems and this curriculum:

1. This is exciting. Mathematics has many really cool relationships.
2. Teach problem solving (the world needs problem solvers).
3. Application - We learn x by applying x to solve y.
4. Engaging in problem solving can enhance conceptual understanding and procedural fluency. The *National Mathematics Panel* said, "For all content areas, conceptual understanding, computational fluency, and problem-solving skills are each essential and mutually reinforcing, influencing performance on such varied tasks as estimation, word problems, and computation." (Final report, page 30).
5. These problems usually bring together (show) the interplay (connections) between algebra and geometry (and more, such as proportional reasoning).
6. Help students learn how to look for and express regularity in repeated reasoning.
7. Higher order thinking. Analysis (breaking into parts), synthesis (combining ideas), evaluation (making an informed judgement).
8. Depth of knowledge is gained and demonstrated by applying knowledge to solve problems. Teach critical thinking. To be successful in college and careers, one must be able to think critically. (The world needs people who can think critically).
9. Help students learn how to use appropriate tools strategically.
10. Help students learn how to look for and make use of structure.
11. Creativity aspect - By combining approaches, methods, and techniques one can find creative solutions to problems (the world needs people who can think creatively).
12. Learning pure ***mathematics as a human endeavor***. Furthermore, some can grow to enjoy mathematics (the world needs math majors and math teachers).

The audience for these problems and this curriculum include:

- High school students who are (a) at least half way through a course in geometry, (b) in advanced algebra, (c) in pre-calculus or (d) calculus.
- Students planning on taking part in a math contest (AMC8, AMC10, AMC12, state contest, college level contest).
- Potential math majors or other STEM majors (the world needs STEM majors).
- College math majors who plan to be high school teachers.
- *Most importantly,* any person who may, at some point, find they have a predilection for mathematics.

Handout for ISMAA 2019– SIU-C
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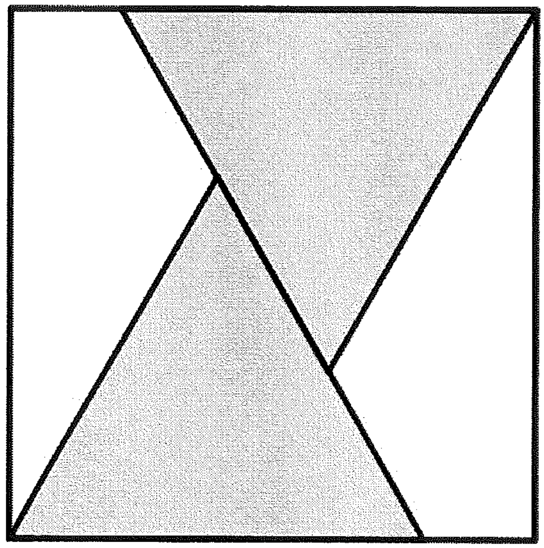
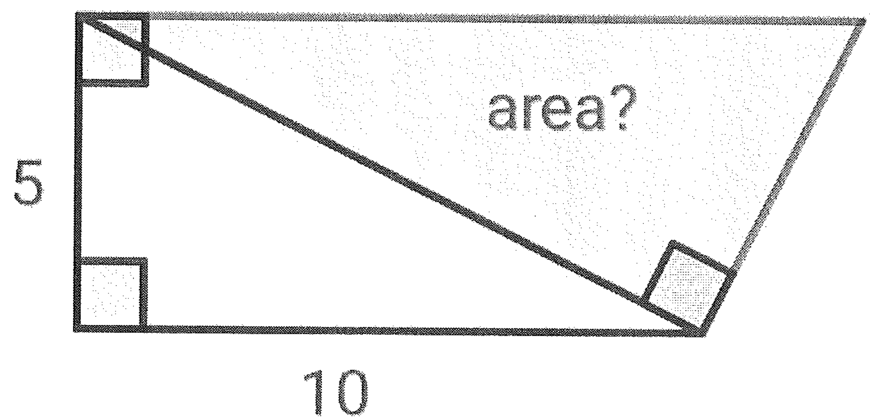
Two Equilaterals in a Square

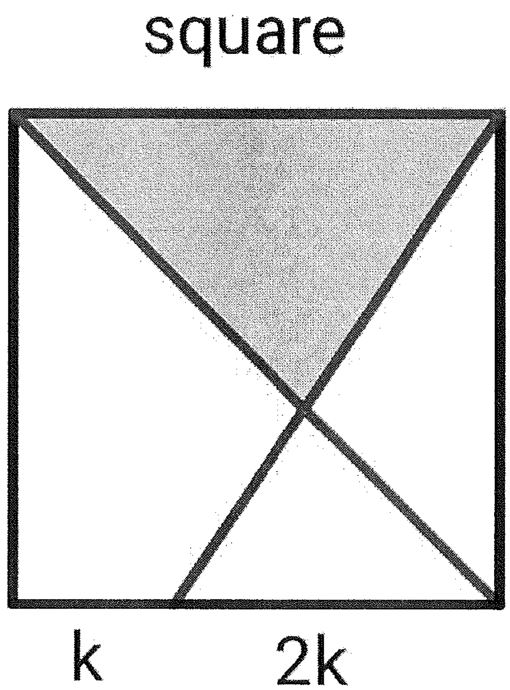
Two equilateral triangles in a square.
Is more or less than half the square shaded?
(What is the exact ratio?)



Instant Classic 1



Fractional Triangle in a Square

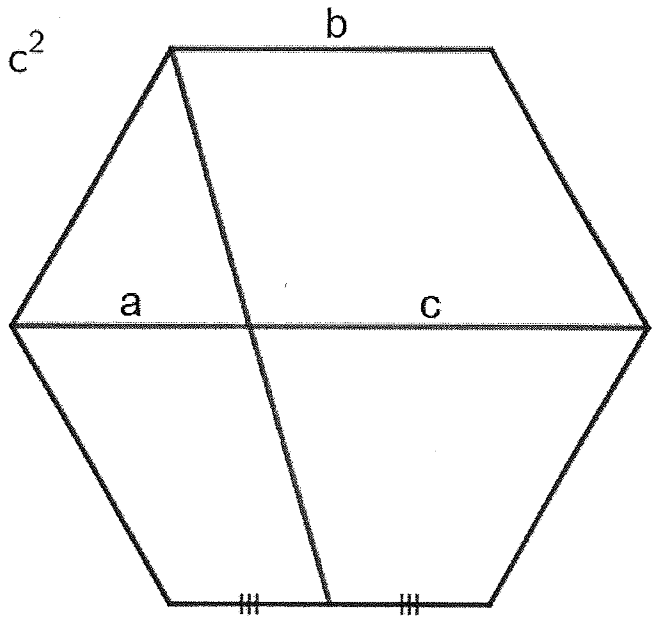


What fraction is shaded?

Pythagorean in a Hexagon

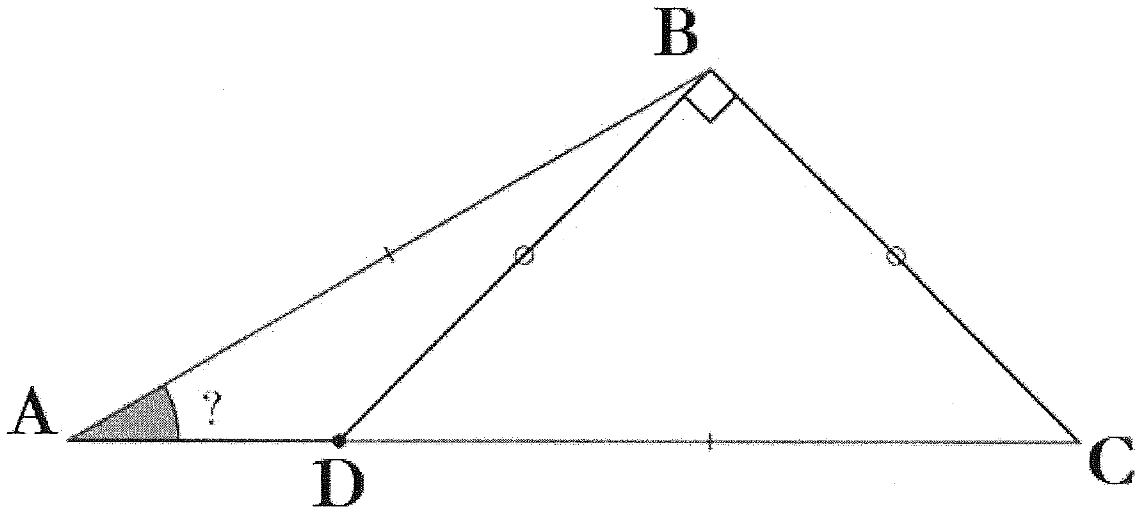
Regular hexagon

Show : $a^2 + b^2 = c^2$



Obtuse Laying on an Isosceles Right Triangle

AB = CD. BD = BC. Find m∠A.

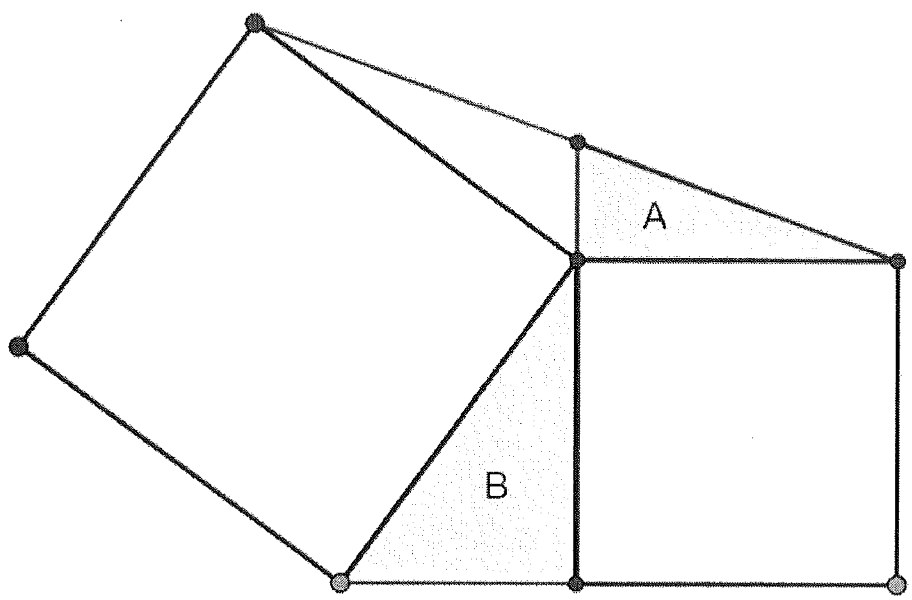


Source: https://twitter.com/puzzlist/status/1093042288886837249

Two Squares and Two Right Triangular Regions

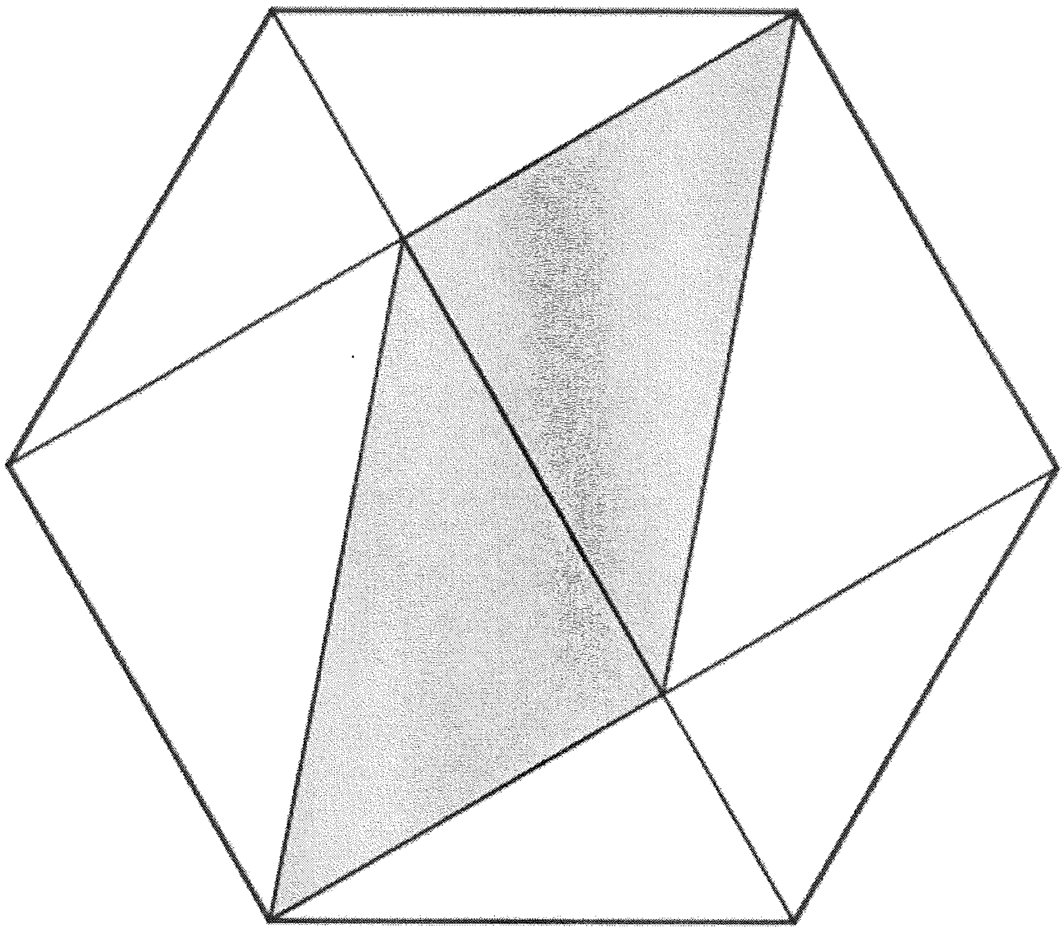
Two squares and two right triangular regions.

Find the ratio of the areas: $\dfrac{A}{B}$



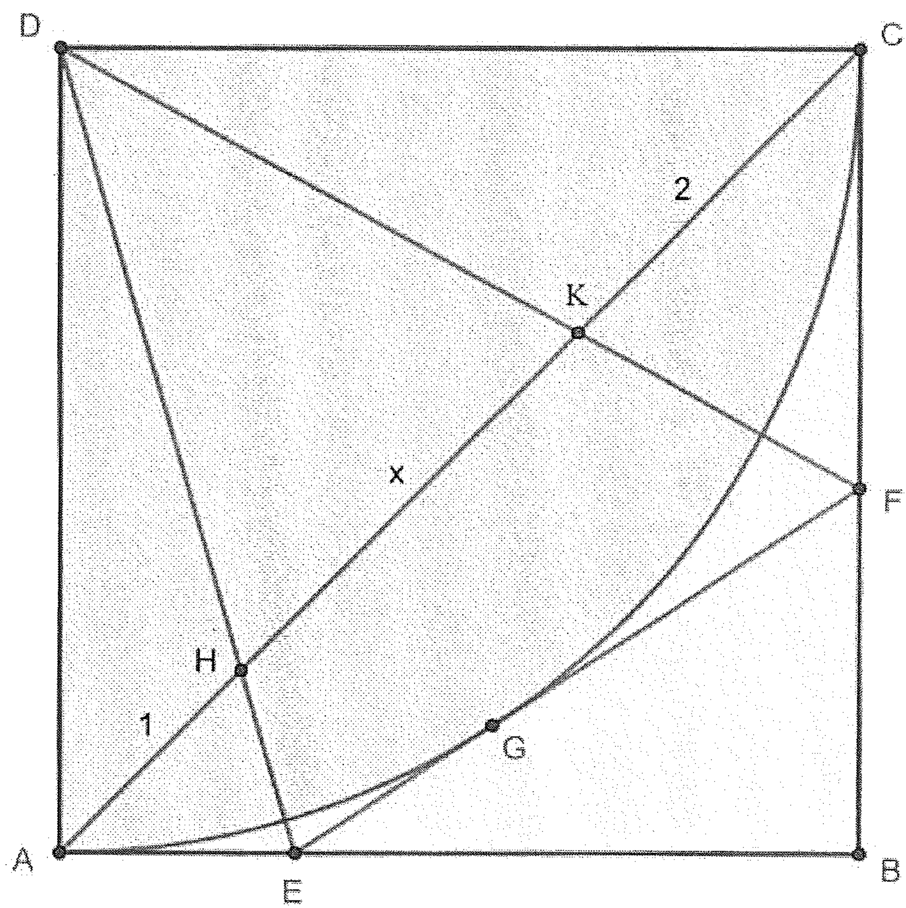
Parallelogram in a Hexagon

In this regular hexagon, three diagonals have been drawn to form this parallelogram. What fraction of the hexagon is shaded?



The Big Finish

A quarter circle is inscribed in a square ABCD with diagonal $\overline{AC}$. Point G is on the circle. The tangent through G meets the square in points E and F.
Draw segments $\overline{DE}$ and $\overline{DF}$. This defines H and K.
AH = 1, CK = 2. Find HK.



Source: https://twitter.com/JhuriaMikki/status/1096991539291578370

Printed: 3/22/18

Additional Geometry* Problems Using the Structure-Symmetry-Similarity Strategy

*You'll use algebra.

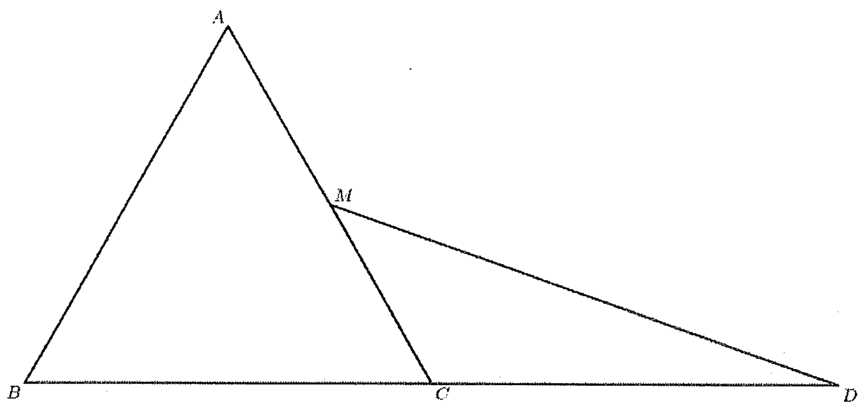
Source for AMC 10 and AMC 12 Problems: Art of Problem Solving – AoPS Wiki

https://artofproblemsolving.com/wiki/index.php?title=Main_Page (many more online!)

1. AMC 10 2005B #14

Problem 14

Equilateral $\triangle ABC$ has side length 2, M is the midpoint of $\overline{AC}$, and C is the midpoint of $\overline{BD}$. What is the area of $\triangle CDM$?



(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{3}{4}$ (C) $\frac{\sqrt{3}}{2}$ (D) 1 (E) $\sqrt{2}$

2. AMC 10 2005B #23

Problem 23

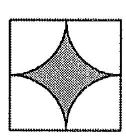
In trapezoid $ABCD$ we have $\overline{AB}$ parallel to $\overline{DC}$, E as the midpoint of $\overline{BC}$, and F as the midpoint of $\overline{DA}$. The area of $ABEF$ is twice the area of $FECD$. What is AB/DC?

(A)2 (B)3 (C)5 (D)6 (E)8

3. AMC 10 2005B #8

Problem 8

An 8-foot by 10-foot floor is tiled with square tiles of size 1 foot by 1 foot. Each tile has a pattern consisting of four white quarter circles of radius $1/2$ foot centered at each corner of the tile. The remaining portion of the tile is shaded. How many square feet of the floor are shaded?

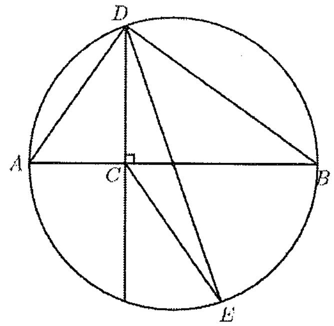


(A) $80 - 20\pi$ (B) $60 - 10\pi$ (C) $80 - 10\pi$ (D) $60 + 10\pi$ (E) $80 + 10\pi$

4. AMC 10 2005A #23

Problem 23

Let AB be a diameter of a circle and let C be a point on AB with $2 \cdot AC = BC$. Let D and E be points on the circle such that $DC \perp AB$ and DE is a second diameter. What is the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$?



(A) $\frac{1}{6}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) $\frac{2}{3}$

5. AMC 10 2005A #25

Problem 25

In ABC we have $AB = 25$, $BC = 39$, and $AC = 42$. Points D and E are on AB and AC respectively, with $AD = 19$ and $AE = 14$. What is the ratio of the area of triangle ADE to the area of the quadrilateral $BCED$?

(A) $\frac{266}{1521}$ (B) $\frac{19}{75}$ (C) $\frac{1}{3}$ (D) $\frac{19}{56}$ (E) 1

6. AMC 12 2015B #13

Problem 13

Quadrilateral $ABCD$ is inscribed in a circle with $\angle BAC = 70°$, $\angle ADB = 40°$, $AD = 4$, and $BC = 6$. What is AC?

(A) $3 + \sqrt{5}$ **(B)** 6 **(C)** $\frac{9}{2}\sqrt{2}$ **(D)** $8 - \sqrt{2}$ **(E)** 7

7. AMC 12 2015B #16

Problem 16

A regular hexagon with sides of length 6 has an isosceles triangle attached to each side. Each of these triangles has two sides of length 8. The isosceles triangles are folded to make a pyramid with the hexagon as the base of the pyramid. What is the volume of the pyramid?

(A) 18 **(B)** 162 **(C)** $36\sqrt{21}$ **(D)** $18\sqrt{138}$ **(E)** $54\sqrt{21}$

8. AMC 12 2015B #19

Problem 19

In $\triangle ABC$, $\angle C = 90°$ and $AB = 12$. Squares $ABXY$ and $ACWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. What is the perimeter of the triangle?

(A) $12 + 9\sqrt{3}$ **(B)** $18 + 6\sqrt{3}$ **(C)** $12 + 12\sqrt{2}$ **(D)** 30 **(E)** 32