The Opportunity Cost of Holding a “Naïve” Portfolio

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Abstract

The paper explores the effect of “naïve” portfolio strategies on investors’ welfare. A “naïve” portfolio as a sub-optimal investment strategy produces sub-optimal asset allocations that result in investors’ welfare losses. To measure those losses I compare sub-optimal portfolios with optimal portfolios using the proportionate opportunity cost with various CRRA utility functions. A vector autoregression is used to generate the joint distribution of asset returns. I show that the opportunity cost of investing in “naïve” portfolios does not exceed 16.7% while investing in the optimal number of asset and does not exceed 20.4% while investing in a sub-optimal number of assets.

JEL classification number: G11

Keywords: probability distribution function of stock returns; proportionate opportunity cost; optimal portfolio strategy; investors’ welfare losses

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I. Introduction

“Naïve” portfolios are portfolios in which \((1/n)\)th of initial wealth of an investor is invested in each of \(n\) assets (Kroll et al. (1984)). The “naïve” portfolio strategy does not involve any optimization procedure and, therefore, choosing a “naïve” portfolio instead of a well-diversified results in a welfare loss for an investor. How large those welfare losses can be from such a sub-optimal asset allocation?


Although these studies provide an extensive catalogue of factors that affect household portfolio choice, they do not address the impact on household-investors’ utility of a sub-optimal asset allocation when initial wealth is evenly allocated between \(n\) assets.

Are well-diversified optimal portfolios substantially different from any sub-optimal, including “naïve”, portfolios?
Cheng and Liang (2000) found that there is evidence to support the idea that optimally diversified portfolios are more efficient than sub-optimally diversified portfolios in the context of mean-variance framework. To test the efficiency difference they compared the Sharpe ratio for a well-diversified portfolio with the Sharpe ratio of a sub-optimally diversified portfolio.

Brennan and Torous (1999) have addressed the issue of the cost of sub-optimal diversification, and Fama (1972) and Sankaran and Patil (1999) have addressed the issue of how many securities is enough for a well-diversified portfolio by using the $\frac{1}{n}$ rule, “naïve” strategy, for their portfolios. Bender et al. (2010) found that equally-weighted naïve portfolios produce similar returns to a traditional 60/40 (equity/bonds) allocation.

While comparing naïve portfolio strategy and mean-variance portfolios Duchin and Levy (2009), DeMiguel, Garlappi, and Uppal (2009), and Fugazza, Guidolin, and Nicodano (2010) found that for smaller portfolios naïve investment strategy is more efficient and for larger portfolios the mean-variance strategy is more efficient (in terms of estimation error); in terms of Sharpe ratio and certainty equivalent naïve portfolios outperform mean-variance portfolios; and investors with investment horizon of one year and longer should expect, on average, higher returns from optimized portfolios.

Although these papers may explain either the presence or absence of certain types of asset in household portfolios or the optimal number of assets that maximizes the expected portfolio return, the question that remains unanswered is: how costly is it to use the “naïve” portfolio strategy?

Estimating utility losses for an investor who invests equal shears of initial wealth into $n$ assets is important for analyzing portfolio allocation decisions.
The best way to approach the problem of estimating the cost of investing in naïve portfolios is to calculate the proportionate opportunity cost of sub-optimal (naïve) asset allocation while using optimal portfolios (not mean-variance efficient as Duchin and Levy (2009), DeMiguel, Garlappi, and Uppal (2009), and Fugazza, Guidolin, and Nicodano (2010) did, but rather globally optimal) that can be found through an optimization procedure.

The procedure followed in this article for calculating the proportionate opportunity cost includes random asset selection for investors’ portfolios, estimation of a Vector AutoRegressive (VAR) process, derivation of the joint probability distribution function of asset returns, and calculation of optimal constrained and unconstrained portfolios.

In this article I show that with a nominally risk-free asset “naïve” portfolio strategy performs almost as well as the unconstrained portfolio strategy for investors with high levels of risk aversion only. There is a considerable cost to incur for investors of low and medium risk aversion (up to 20.4% of initial wealth if restricted to 9 (nine) assets only). The cost becomes even larger when extreme returns in the original historical data set are exaggerated.

This article is organized as follows. Section II introduces the concept of the proportionate opportunity cost, Section III describes the procedure of random asset selection for investors’ portfolios, of inferring the joint probability distribution function of asset returns, of computing constrained and unconstrained optimal portfolios and of calculating the proportionate opportunity cost. Section IV discusses the results of the study and Section V concludes.
II. Proportionate opportunity cost.

In order to measure welfare losses from investing in constrained, “naïve”, portfolios, I compare expected utility from portfolio constrained to have equal dollar-shares, with that from the optimal unconstrained portfolio where dollar-shares are found through an optimization procedure. The comparison is done by using the concept of opportunity cost as developed by Brennan and Torous (1999) and Tew, Reid and Witt (1991).

Proportionate opportunity cost is the best way to measure investors’ welfare losses because results are readily interpretable as intuitively “large” or “small”, which would not be true if compensating payments were expressed in additive dollar terms.

Under the assumption of the constant relative risk aversion (CRRA) utility function:

\[
U(\tilde{w}) = \begin{cases} 
\frac{1}{\gamma} \tilde{w}^\gamma, & \gamma < 1, \gamma \neq 0, \tilde{w} > 0 \\
-\infty, & \tilde{w} \leq 0 
\end{cases}
\]

where \(U(\tilde{w})\) is the utility from final wealth, \(\tilde{w}\) is final wealth, and \(1-\gamma\) is the level of relative risk aversion. Then, willingness to accept payment to accept the constraint, \(\theta-1.0\), is defined by:

\[
E U(\theta w_0 \tilde{R}^c) = E U(w_0 \tilde{R}^u)
\]

where \(w_0\) is initial wealth, \(\tilde{R}^u\) is the gross portfolio return (per dollar invested) from optimally investing without the constraint implied by the naïve portfolio strategy, \(\tilde{R}^c\) is the gross portfolio return from investing in naïve portfolios, \(U(\theta w_0 \tilde{R}^c)\) is utility from final wealth when the investor is being constrained and compensated, and \(U(w_0 \tilde{R}^u)\) is utility from final wealth when the investor is unconstrained and uncompensated.
Therefore, in the case of the CRRA utility (2):

\[
E\left(\frac{1}{\gamma} (\theta w_0 \tilde{R}^\gamma)\right) = E\left(\frac{1}{\gamma} (w_0 \tilde{R}^\gamma)\right)
\]

Solving (2) and (3) with the utility function (1) gives:

\[
\theta = \left[ \frac{E(\tilde{R}^\gamma)_{\text{unconstrained}}}{E(\tilde{R}^\gamma)_{\text{constrained}}} \right]^{\frac{1}{\gamma}}
\]

Then \(\theta - 1.0\) shows how much the investor will require to get paid as a fraction of initial wealth in order to accept the constrained portfolio (willingness to accept the constraint).

The fraction of initial wealth that the investor will be willing to pay to stay with the unconstrained portfolio (the willingness to pay to avoid the constraint), \(1.0 - \phi\), is defined by:

\[
E\ U(w_0 \tilde{R}^\gamma) = E\ U(\phi w_0 \tilde{R}^\gamma)
\]

where \(U(w_0 \tilde{R}^\gamma)\) is the utility from the final wealth when the investor has the constrained portfolio, and \(U(\phi w_0 \tilde{R}^\gamma)\) is the utility from the final wealth when the investor pays the fraction \(1.0 - \phi\) of initial wealth in order to avoid being constrained. Therefore, in case of the CRRA utility function, (1):

\[
E\left(\frac{1}{\gamma} (w_0 \tilde{R}^\gamma)\right) = E\left(\frac{1}{\gamma} (\phi w_0 \tilde{R}^\gamma)\right)
\]

Solving (5) and (6) for \(\phi\) gives:
\[
\varphi = \left[ \frac{E(\tilde{R}^\gamma)_{\text{constrained}}}{E(\tilde{R}^\gamma)_{\text{unconstrained}}} \right]^{\frac{1}{\gamma}}
\]

But (7) is the exact reciprocal of (4): \(\varphi = \frac{1}{\theta}\). In other words, the proportion of initial wealth that the investor will have to be paid in order to stay with the constrained portfolio \((\theta - 1.0)\) can be easily converted into the proportion of initial wealth that the investor will be willing to give up to stay with the unconstrained portfolio \((1.0 - \varphi)\).

The last appropriate measure of the proportionate opportunity cost is the ratio of the certainty equivalents. The certainty equivalent shows the amount of certain wealth that would be viewed with indifference by an investor relative to having an uncertain amount of wealth. Consider the ratio of the certainty equivalents of the unconstrained and constrained portfolios.

Let us denote the certainty equivalent by CE; then, using the utility function (1) two expressions follow: the certainty equivalent for the investor with the unconstrained portfolio (8) and the certainty equivalent for the investor with the constrained portfolio (9):

\[
\gamma \frac{1}{\gamma} \text{CE}_{\gamma, \text{unconstrained}} = \frac{1}{\gamma} w^*_\gamma E(\tilde{R}^\gamma)_{\text{unconstrained}}
\]

(8)

\[
\gamma \frac{1}{\gamma} \text{CE}_{\gamma, \text{constrained}} = \frac{1}{\gamma} w^*_\gamma E(\tilde{R}^\gamma)_{\text{constrained}}
\]

(9)

Solving for the ratio of the certainty equivalents for the unconstrained and constrained portfolios with (8) and (9) gives:

\[
\frac{CE_{\text{unconstrained}}}{CE_{\text{constrained}}} = \left[ \frac{E(\tilde{R}^\gamma)_{\text{unconstrained}}}{E(\tilde{R}^\gamma)_{\text{constrained}}} \right]^{\frac{1}{\gamma}}
\]

(10)
The right hand side of (10) is identical to the formula for \( \theta \) (see (4)).

Under CRRA (1), one plus the proportionate willingness to accept payment and the ratio of the certainty equivalents of unconstrained to constrained optimal portfolios are equal. Since the ratio of the certainty equivalents is unitless and in particular has no time units, the proportionate opportunity cost, \( \theta - 1.0 \), is timeless. But its value depends on a number of months to investment horizon. For example, with the investment horizon of one month if the unconstrained certainty equivalent is 1.08 and the constrained certainty equivalent is 1.03, then the proportionate willingness to accept payment will be \((1.08/1.03)\). But with a horizon of \( T \) months the unconstrained certainty equivalent will be \(1.08^T\) and the constrained certainty equivalent will be \(1.03^T\). Then the ratio of the certainty equivalents is \((1.08/1.03)^T\). Therefore, the proportionate willingness to accept payment, \( \theta - 1.0 \), is a recurring cost.

III. The Procedure

This section of the paper describes the procedure of forming investors’ portfolios, of inferring the joint probability distribution function of asset returns via a vector autoregression, of computing the constrained optimal and unconstrained optimal portfolios, and of calculating of the proportionate opportunity cost.

1. Portfolio Formation

The data set used is monthly historically occurring asset returns over the ten-year period from January 1998 through December 2007. With different time units the conclusions might be affected. With quarterly asset returns I would need to extend the
time period to 1978.I to 2007.IV, and with annual asset returns the new time period will be from 1888 to 2007 just to get the same 120 data points. In both cases I would have to deal with very old asset returns that might not accurately reflect the true probability distribution facing current investors. The choice of monthly time units is also consistent with the studies of Simaan (1993) and Kroll, Levy, and Markowitz (1984).

To form the unconstrained portfolio I use \( n \)-1 risky assets with Treasury bills as the nominally risk-free asset. The number \( n \), as optimal number of assets, changes as the degree of risk aversion changes\(^1\).

The constrained portfolio includes the same set of risky assets and Treasury bills as the nominally risk-free asset.

2. Vector Autoregression of Returns

To get expected values and probability distributions of real returns for the \( n \)-1 risky assets and Treasury bills at time \( T+1 \), the portfolio formation period, I estimate a vector autoregressive process (VAR). Then I derive the joint probability distribution for the \( n \)-1 risky assets and Treasury bills real returns. Finally, I construct optimal constrained and optimal unconstrained portfolios.

To derive the joint probability distribution of empirical deviations from the VAR-estimated conditional means for those \( n \)-1 assets returns and inflation, the following methodology is applied: The nominal return on asset \( i \) at time \( t \) minus the nominal return on Treasury bills at time \( t \) gives the excess return on asset \( i \) at time \( t \) \((x_{i,t})\) for \( i=1,\ldots,4 \) and

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\(^1\) Melkumian, Alla A. “The Opportunity Cost of Being Constrained by the Number of Assets in Investors’ Portfolios”, 2007, working paper
for \( t=1, \ldots, T \). Running a VAR for excess returns of those \( n-1 \) assets and realized inflation, as

\[
\begin{bmatrix}
    x_{1,t} \\
    \vdots \\
    x_{n-1,t} \\
    \pi_t
\end{bmatrix}
= \begin{bmatrix}
    c_1 \\
    \vdots \\
    c_{n-1} \\
    c_n
\end{bmatrix}
+ \begin{bmatrix}
    v_{1,1}(L) & \cdots & v_{1,n}(L) \\
    \vdots & \ddots & \vdots \\
    v_{n-1,1}(L) & \cdots & v_{n-1,n}(L) \\
    v_{n,1}(L) & \cdots & v_{n,n}(L)
\end{bmatrix}
\begin{bmatrix}
    x_{1,t} \\
    \vdots \\
    x_{n-1,t} \\
    \pi_t
\end{bmatrix}
+ \begin{bmatrix}
    \varepsilon_{1,t} \\
    \vdots \\
    \varepsilon_{n-1,t} \\
    \varepsilon_{\pi,t}
\end{bmatrix},
\]

(11)

\( \{ \hat{c}_i \}, \{ \hat{\varepsilon}_{i,t} \} \) and \( \{ \hat{v}_{i,k}(L) \} \) are obtained, where

\[
\hat{v}_{i,k}(L) = \hat{\delta}_{i,k} L^1 + \hat{\delta}_{i,k}^2 L^2 + \ldots
\]

(12)

The vector of conditional expected values of excess returns for time \( T+1 \) and expected inflation for time \( T+1 \) is computed as:

\[
\begin{bmatrix}
    E_T x_{1,T+1} \\
    \vdots \\
    E_T x_{n-1,T+1} \\
    E_T \pi_{T+1}
\end{bmatrix}
= \begin{bmatrix}
    \hat{c}_1 \\
    \vdots \\
    \hat{c}_{n-1} \\
    \hat{c}_n
\end{bmatrix}
+ \begin{bmatrix}
    \hat{v}_{1,1}(L) & \cdots & \hat{v}_{1,n}(L) \\
    \vdots & \ddots & \vdots \\
    \hat{v}_{n-1,1}(L) & \cdots & \hat{v}_{n-1,n}(L) \\
    \hat{v}_{n,1}(L) & \cdots & \hat{v}_{n,n}(L)
\end{bmatrix}
\begin{bmatrix}
    x_{1,T+1} \\
    \vdots \\
    x_{n-1,T+1} \\
    \pi_{T+1}
\end{bmatrix}
+ \begin{bmatrix}
    \hat{\varepsilon}_{1,T+1} \\
    \vdots \\
    \hat{\varepsilon}_{n-1,T+1} \\
    \hat{\varepsilon}_{\pi,T+1}
\end{bmatrix},
\]

(13)

The expected real return on index \( i \) in period \( T+1 \), the portfolio formation period, is

\[
\begin{bmatrix}
    E_T r_{1,T+1} \\
    \vdots \\
    E_T r_{n-1,T+1}
\end{bmatrix}
= \begin{bmatrix}
    E_T x_{1,T+1} \\
    \vdots \\
    E_T x_{n-1,T+1}
\end{bmatrix}
+ \begin{bmatrix}
    r_{TB,T+1}^n \\
    \vdots \\
    r_{TB,T+1}^n
\end{bmatrix}
- \begin{bmatrix}
    E_T \pi_{T+1}
\end{bmatrix},
\]

(14)

where \( r_{TB,T+1}^n \) is the ex ante observed nominal return on Treasury bills for time \( T+1 \). The expected real return on Treasury bills for time \( T+1 \) is

\[
E_T r_{TB,T+1} = r_{TB,T+1}^n - E_T \pi_{T+1}.
\]

(15)

Finally, the conditional probability distribution for real returns for time \( T+1 \) is determined by
\[
\begin{bmatrix}
\tilde{\epsilon}_{1,T+1} \\
\vdots \\
\tilde{\epsilon}_{n-1,T+1} \\
\tilde{\epsilon}_{\pi,T+1}
\end{bmatrix} = \begin{bmatrix}
E_T x_{1,T+1} \\
\vdots \\
E_T x_{n-1,T+1} \\
0
\end{bmatrix} + \begin{bmatrix}
r_{TB,T+1}^n \\
\vdots \\
r_{TB,T+1}^n \\
0
\end{bmatrix} - \begin{bmatrix}
E_T \pi_{T+1} \\
\vdots \\
E_T \pi_{T+1} \\
0
\end{bmatrix} + \begin{bmatrix}
\tilde{\epsilon}_{1,T+1} \\
\vdots \\
\tilde{\epsilon}_{n-1,T+1} \\
\tilde{\epsilon}_{\pi,T+1}
\end{bmatrix} - \begin{bmatrix}
\tilde{\epsilon}_{1,T+1} \\
\vdots \\
\tilde{\epsilon}_{n-1,T+1} \\
\tilde{\epsilon}_{\pi,T+1}
\end{bmatrix}
\]

where \( \tilde{\epsilon}_{1,T+1} \) takes on the historically observed values from regression (6), \( t=1,2,\ldots,T \), with equal probabilities (1/T).

This method for deriving asset returns probability distribution functions, using historically occurring innovations to asset returns captured through a VAR procedure, is superior to the VAR method mentioned in the earlier literature, e.g., Campbell and Viceira (2002). The earlier literature on derivation of asset returns probability distribution functions assumes that the distribution of asset returns is static, not evolving over time. But the reality is such that the asset returns distribution is dynamic, depending on both recent realizations and the fixed historical distribution of shocks to the dynamic asset returns process. Thus a better way to derive asset returns probability distribution functions is to include the dynamics of the past history of asset returns.

The probability distribution of returns derived as shown in (11)-(16) is used for both types of portfolios, constrained as well as unconstrained.
3. Constrained Portfolios

Using the information about the \(n-1\) risky assets and Treasury bills’ derived probability distribution of real returns compute equal-share constrained optimal portfolios of \(n\) assets where each portfolio share will be equal \(\frac{1}{n}\) of initial wealth.

4. Unconstrained Portfolios

The next step is to obtain the unconstrained optimal portfolios with \(n-1\) risky assets and Treasury bills as the solution to the problem:

\[
\text{Max}_{\{\alpha_1, \ldots, \alpha_{n-1}\}} \quad \text{EU}(\tilde{\omega}) = \text{Max} \quad E\left\{ \frac{1}{\gamma} \left[ \sum_{i=1}^{n} \alpha_i \tilde{r}_i + \sum_{j=1}^{n-1} \alpha_{j-1} \tilde{r}_{j-1} + (1 - \sum_{j=1}^{n-1} \alpha_{j-1}) \tilde{r}_{TB} \right]^\gamma \right\}
\]

where \(\alpha_1, \ldots, \alpha_{n-1}\) are the \(n-1\) individual assets’ portfolio shares in the unconstrained optimal portfolio. To get the portfolio, search over the \((\alpha_1, \ldots, \alpha_{n-1})\) space to optimize expected utility, again using nonlinear optimization by a quasi-Newton method based on using iterative solutions of the first-order conditions of problem (17). The expectation is taken over the joint probability distribution derived from the \(n\)-asset VAR.

5. Calculating Opportunity Cost

On the basis of the constrained and unconstrained optimal portfolios obtained above, now calculate the proportionate opportunity cost, \(\theta-1\). The following notation is used: \(E(\tilde{R}^\gamma)^{OS}\) (where \((\tilde{R}^\gamma)^{OS}\) is the gross return for the optimal unconstrained portfolio
with optimal shares), and \( E(\tilde{R}^{\gamma})^{ES} \) (where \((\tilde{R}^{\gamma})^{OS}\) is the gross returns for the optimal constrained portfolios with equal shares). \( E(\tilde{R}^{\gamma})^{OS} \) is determined as follows:

\[
(18) \quad E(\tilde{R}^{\gamma})^{OS} = \frac{1}{T} \sum_{t=1}^{T} \left\{ \alpha_{1}^* \cdot \alpha_{n-1}^* \cdot (1-\alpha_{1}^* - \cdots - \alpha_{n-1}^*) \left[ \begin{array}{c}
E_{T_{1,T+1}} + \epsilon_{1,t}^r \\
E_{T_{n-1,T+1}} + \epsilon_{n-1,t}^r \\
E_{T_{TB,T+1}} + \epsilon_{TB,t}^r
\end{array} \right] \right\}^\gamma
\]

where vector \( \alpha_t^* \) is the vector of optimal shares for the unconstrained portfolio (with \( \alpha_n = 1 - \alpha_1 - \cdots - \alpha_{n-1} \)), and the vectors of \( E_{t_{1,T+1}} + \epsilon_{1,t} - \epsilon_{\pi,t} \) and \( E_{t_{TB,T+1}} + 0 - \epsilon_{\pi,t} \) for \( t=1,\ldots,T \) are the vectors of particular possible values of real returns (conditional on the data set for times \( t=1 \) through \( T \)) at time \( T+1 \).

The constrained values are determined as follows:

\[
(19) \quad E(\tilde{R}^{\gamma})^{ES} = \frac{1}{T} \sum_{t=1}^{T} \left\{ \frac{1}{n} \cdot \frac{1}{n} \left[ \begin{array}{c}
E_{T_{1,T+1}} + \epsilon_{1,t}^r \\
E_{T_{n-1,T+1}} + \epsilon_{n-1,t}^r \\
E_{T_{TB,T+1}} + \epsilon_{TB,t}^r
\end{array} \right] \right\}^\gamma
\]

where \( \frac{1}{n} \) are the equal portfolio shares for constrained portfolios.

Finally, find numerical values for \( \theta \):

\[
(20) \quad \theta = \left( \frac{E(\tilde{R}^{\gamma})^{OS}}{E(\tilde{R}^{\gamma})^{ES}} \right)^{\frac{1}{\gamma}}
\]

The above exercise will be done for each of 11 alternative values of the risk aversion parameter, \( \gamma \).
IV. Results

The results of this research project are as follows.

1. Opportunity Costs

1a. Results Derived from Historical Returns Data with No Exaggeration of Extreme Returns.

Table 1 and Table 2 represent the results from calculation of 1,000 values of the proportionate opportunity cost for 11 different levels of relative risk aversion for alternatively the well-diversified number of assets (the number is different for different degrees of risk aversion, and noted in the second column of Table 1 and nine (9) assets (Simaan (1993) argued that 10 stocks will be sufficient to trace the efficient frontier and Tew, Reid and Witt (1991) used from two to nine stocks in their calculations), based on historically occurring asset returns over the 10-year period January 1998 through December 2007.

Of all the values of relative risk aversion examined the lowest mean (over 1,000 replications) of the proportionate opportunity cost for the well-diversified and the nine (9) assets corresponds to the high level of relative risk aversion of 31. The highest mean (over 1,000 replications) of the proportionate opportunity cost for both nine (9) and the well-diversified number of assets corresponds to the lowest level of relative risk aversion of 0.7. This suggests that optimal unconstrained portfolios offer high risk-tolerance

Table 1

The Proportionate Opportunity Cost, \( (\theta - I) \), for Various Values of Relative Risk Aversion for the Well-Diversified Number of Assets
investors broader, more daring investment opportunities than constrained optimal portfolios, and so the investors will require a premium to give up those investment opportunities.

Both tables clearly show that as level of relative risk aversion increases the proportionate opportunity cost decreases, given the CRRA utility function, (1), the better the constrained, “naïve”, portfolio performs and the lower the proportion of initial wealth an investor requires to stay constrained and accept the “naïve” portfolio instead of optimal unconstrained portfolio. This is not surprising. As risk aversion decreases, as investors become more risk tolerant, they consider optimal unconstrained portfolio as their best choice that does not place any restrictions on their investment behavior and let

Table 2

The Proportionate Opportunity Cost, ($\theta - I$), for Various Values of Relative Risk Aversion for 9 Assets

them follow a very aggressive short sale strategy that will not be possible under the constrained portfolio strategy, where equal number of dollars is allocated to different assets. Therefore, investors will require higher proportion of initial wealth as the payment to stay constrained and accept the optimal constrained “naïve” portfolio.

Table 1 and Table 2 also show that as the level of relative risk aversion increases the standard deviations of the proportionate opportunity costs decrease: the distributions of the opportunity cost are getting “tighter”. As the level of relative risk aversion
increases more and more of the numerical values for the opportunity costs are concentrating around their means. Thus we see that as the level of risk aversion increases, as investors become less risk tolerant, the perceptions of the constrained “naïve” portfolio strategy for investors with different asset sets are more similar to each other than perceptions of that strategy for different investors with lower risk aversion.

For the well-diversified number of assets, Table 1 shows that the lowest mean (over 1,000 replications) of the proportionate opportunity cost, 1.0% (0.010), corresponds to the high level of risk aversion of 31. This means that an investor with the level of relative risk aversion of 31 being unconstrained will be equally happy as if he was constrained but had 1.0% more of initial wealth. The highest mean (over 1,000 replications) of the proportionate opportunity cost, 16.7% (0.167), corresponds to the very low level of relative risk aversion of 0.7. This means that an investor with the level of relative risk aversion of 0.7 being unconstrained will be equally happy as if he was constrained but had 16.7% more of initial wealth.

For low levels (from 3 to 0.7) of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost ranges from 13.7% (0.137) for relative risk aversion of 3 to 16.7% (0.167) for relative risk aversion of 0.7. This suggests that even medium risk-tolerance investors value optimal unconstrained portfolios high enough as opposed to constrained investment behavior to require from 1.1% to 1.4% of additional initial wealth to stay constrained.
For high (from 31 to 29) levels of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost is 1.0% (0.010). This suggests that even investors with high degree of risk aversion would prefer the optimal portfolio strategy over the “naïve” portfolio strategy. Given the investors’ optimal portfolios at those degree of risk aversion consist of three assets, the “naïve” portfolio strategy will place a third of investors’ initial wealth in each asset, whereas the optimal portfolio strategy will probably short holdings of the two risky assets and will suggest a long holding on the risk-free asset, a treasury bill. And as the result, deprived of shorting risky assets, investors will demand a reward for submitting to the sub-optimal “naïve” portfolio strategy.

For nine (9) assets, Table 2 shows that the lowest mean (over 1,000 replications) of the proportionate opportunity cost, 1.0% (0.010), corresponds to the high level of risk aversion of 31. This means that an investor with the level of relative risk aversion of 31 being unconstrained will be equally happy as if he was constrained but had 1.0% more of initial wealth. The highest mean (over 1,000 replications) of the proportionate opportunity cost, 20.4% (0.204), corresponds to the very low level of relative risk aversion of 0.7. This means that an investor with the level of relative risk aversion of 0.7 being unconstrained will be equally happy as if he was constrained but had 20.4% more of initial wealth.

The highest values of the proportionate opportunity cost (the means over 1,000 replications) correspond to low levels of relative risk aversion (from 0.7 to 3) and range from 5.2% (0.052) for relative risk aversion of 3 to 20.4% (0.204) for relative risk aversion of 0.7. Investors with low levels of risk aversion (from 3 to 0.7) in the presence
of nine (9) assets will require from 5.2% to 20.4% of initial wealth to stay constrained and accept the “naïve” constrained optimal portfolio. These magnitudes, as a matter of fact, are higher than those for the well-diversified number of assets in Table 1. This suggests that even nine (9) assets are considered to be large enough to trace investors’ behavior it is not large enough to call a nine-asset portfolio a well-diversified. Therefore, the presence of two constraints: the restrictions on the number of assets and the allocation of initial wealth between those assets produce a larger opportunity cost. And the opportunity cost of 20.4% for investors with degree of risk aversion of 0.7, which is huge, clearly signifies that nine (9) assets is not optimal number of assets for those investors. If more than nine (9) assets available for investment, Table 1, investors find broader and more daring investment strategies that are further away from the “naïve” one, and correspond to a higher proportionate opportunity cost.

For medium (from 12 to 9) levels of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost ranges from 1.2% (0.012) for relative risk aversion of 12 to 1.8% (0.018) for relative risk aversion of 9. These numbers are about the same size as those for the well-diversified number of assets. The almost identical numbers suggest that the constrained “naïve” investment strategies are as costly to the well-diversified investors as for investors with nine-asset portfolios. It can be explained by the fact that at degree of risk aversion of 9 and higher the optimal number of assets is nine and less. Therefore, I expect that the opportunity cost for investing in “naïve” portfolios should be virtually the same for the Table 1 and Table 2 starting from the degree of risk aversion of 9.
For high (from 31 to 29) levels of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost is 1.0% (0.010). This supports my expectations about the opportunity cost being identical for the both tables.

My findings suggest that the more assets are available for investors the further away low risk aversion investors will go from the “naïve” constrained strategy.

2. Optimal Portfolio Shares.

Tables 3 and 4 show typical optimal portfolio shares for unconstrained and constrained portfolio strategies for three different levels of relative risk aversion: low (of 0.7), medium (of 11), and high (of 31) for the well-diversified number of assets and for 9 assets; in each case for a different set of available assets giving an opportunity cost that is typical for that level of risk aversion.

Table 3

Illustrative optimal portfolio shares for unconstrained portfolios and naïve (constrained) portfolios for different values of relative risk aversion for the well-diversified number of assets

In both tables for unconstrained portfolios corresponding to a risk aversion of 0.7, more than 100% of initial wealth, \( w_0 \), is held in the nominal risky assets (asset #1 through asset #23 in Table 3 and asset #1 through assets #8 in Table 4) as a group, and Treasury bills are held in negative quantities.

Table 4

Illustrative optimal portfolio shares for unconstrained portfolios and naïve (constrained) portfolios for different values of relative risk aversion for nine assets
As risk aversion increases, the proportion of initial wealth held in Treasury bills first reaches 0, and then becomes positive; and correspondingly the proportion of initial wealth held in the group of nominally risky assets decreases.

The tables show that unconstrained optimal portfolio shares are not similar for different levels of risk aversion. As a matter of fact, optimal unconstrained portfolios for the low level of relative risk aversion of 0.7 have more extreme quantities (negative as well as positive) of assets than optimal unconstrained portfolios of medium and high levels of relative risk aversion. Extremely negative quantities of assets for high risk-tolerance investors mean that the investors follow a very aggressive short-sale strategy.

Table 3 shows that constrained optimal portfolio shares are not similar for different levels of risk aversion also. The well-diversified number of assets is different for different levels of risk aversion: 24 assets for risk aversion of 0.7, seven assets for risk aversion of 11, and three assets for risk aversion of 31. Given that naïve portfolio places equal shares of initial wealth into \( n \) assets, the constrained portfolio shares are calculated as \( \frac{1}{n} : \frac{1}{24} \) for risk aversion of 0.7, \( \frac{1}{7} \) for risk aversion of 11, and \( \frac{1}{3} \) for risk aversion of 31.

In the case of nine assets, Table 4, portfolio shares are identical across the three levels of risk aversion (0.7, 9, and 31) and are calculated as \( \frac{1}{9} \).

Tables 3 and 4 also show monthly expected returns on unconstrained and constrained portfolios, \( E(X^* R) \), for the three levels of relative risk aversion.

The expected returns for constrained and unconstrained optimal portfolios for risk aversion of 0.7 are very dramatic for the well-diversified portfolios and 9-asset portfolios.
Expected returns are of large size for risk aversion of 11 and of small size for risk aversion of 31. Such extreme magnitudes of expected portfolio returns for high risk-tolerance investors confirm the previously made conclusion about very aggressive short-sale strategies. These magnitudes represent highly leveraged portfolios. For investor with risk aversion of 11 and 31 there is, too, some short-selling going on, but not as aggressive as for investors with risk aversion of 0.7. The less aggressive short-selling for medium- or high risk aversion leads to lower mean portfolio returns.

The big difference between the expected portfolio returns with the well-diversified number of assets and those with 9 assets for the risk aversion of 0.7 is due to the level of diversification, which is optimal in the first case and sub-optimal in the second.

In comparing unconstrained expected portfolio returns and constrained expected portfolio returns for the three levels of risk aversion for the two tables, I find that unconstrained and constrained expected portfolio returns for risk aversion of 31 are close; for risk aversion of 11, they are somewhat close; but for risk aversion of 0.7 they are not close at all. These numbers show that as risk aversion increases, the expected returns on unconstrained and constrained portfolios are closer to each other, and thus an investor is more nearly indifferent between the unconstrained and constrained portfolio strategies. This is due to the fact that as risk aversion increases, investors become less and less involved in short-selling and their portfolio shares become less negative and more positive, and optimal unconstrained and constrained portfolio strategies start resemble each other.
Tables 3 and 4 also show the Certainty Equivalents (CE) calculated for the same three levels of relative risk aversion (0.7, 11, and 31). The CE is defined by

\[ \frac{1}{\gamma} \gamma^1 \gamma = \frac{1}{\gamma} w_0 E(\tilde{R}^\gamma) \]

And so, with \( w_0 = 1 \),

\[ CE^\gamma = \left( E(\tilde{R}^\gamma) \right)^\frac{1}{\gamma} \]

The CE represents the amount of certain wealth that would be viewed with indifference to the optimal portfolio. It is computed for investors of different levels of risk aversion: low (of 0.7), medium (of 11), and high (of 31). The two tables show that as risk aversion increases, the value of the CE decreases (for the unconstrained and constrained portfolio strategies). This suggests that as investors become more afraid of risk, they use less risky portfolio strategies and will be expecting lower returns from those portfolios and, hence, the certain amount of wealth that they will be willing to accept with indifference will decrease.

Large negative and positive asset holdings (Tables 3 and 4) in unconstrained and to some extent in constrained portfolios for investors with the level of risk aversion of 0.7 suggest that the investors take a lot of risk. This raises the question: if the worst possible portfolio outcome occurs, then how much will the investors suffer from such an outcome? It is possible to measure the investors’ proportionate regret from the worst-case scenario with such a risky portfolio.

Tables 5 and 6 show the proportionate regret, \((\theta - 1)\), for the well-diversified number of assets and for 9 assets, that will be incurred by investors if the worst possible outcome of asset returns occurs. This \( \theta \) is defined by
where \( X^* \) is the optimal portfolio, \((X^* R)^{\text{worst}}\) is the one of the 120 states of nature giving the lowest portfolio return, \( U((X^* R)^{\text{worst}}) \) is an investor’s utility from getting the worst possible portfolio outcome and \( EU(X^* R) \) is an investor’s \textit{ex ante} expected utility.

Table 5

The ex post proportionate regret, \((\theta - 1)\), under the worst portfolio outcome for the well-diversified number of assets.

Table 6

The ex post proportionate regret, \((\theta - 1)\), under the worst portfolio outcome for nine assets.

For the unconstrained investors (for the case with the well-diversified number of assets and for nine (9) assets), the mean of the proportionate regret (over 1,000 replications) is the highest for the low level of risk aversion of 0.7 and the lowest for the high level of risk aversion of 31. These numbers show that high risk-tolerance investors choose very risky unconstrained asset allocations. And the allocations become riskier as the number of assets in portfolios reaches the well-diversified level. These asset allocations are so risky at the level of risk aversion of 0.7 that if the worst possible outcome occurs it would require for investors with the well-diversified number of assets to receive 991.9% of initial wealth in compensation and for investors with 9 (nine) assets to receive 920.6% of initial wealth in order to obtain the same level of \textit{ex post} utility as their \textit{ex ante} expected utility. For the high level of risk aversion of 31, the mean of the proportionate regret (over 1000 replications) is 2.2% (0.022) for investors with well-diversified number of assets and 2.7% (0.027) for investors with 9 (nine) assets. Such low
proportionate regret suggests that low risk-tolerance unconstrained investors choose very conservative unconstrained portfolios. So conservative are their allocations that even the worst possible outcome will require for them slightly more than 2% of initial wealth to achieve the same level of utility as their ex ante expected utility.

For constrained portfolio strategies the mean proportionate regret (over 1000 replications) ranges from 20.2% (0.202) for risk aversion of 0.7 to 14.2% (0.142) for risk aversion of 31 for investors with the well-diversified number of assets, and from 19.4% (0.194) for risk aversion of 0.7 to 12.9% (0.129) for risk aversion of 31 for investors with 9 (nine) assets. These numbers show that constrained portfolios have a very restrictive character and do not allow high risk-tolerance investors to take nearly as much risk as they would in the absence of the constraint. For medium and low risk-tolerance investors, constrained portfolios are costlier (the proportionate regret is larger for both well-diversified and for 9 (nine) asset portfolios) than unconstrained. It might be due to the fact that constrained portfolios place equal shares of initial wealth in all assets whereas when investors are unconstrained they would significantly short (medium risk aversion) their Treasury bill holdings or place significantly large portions of their initial wealth in it (high risk aversion) (see Tables 3 and 4). Therefore, the portfolio share constraint would be especially binding for investors of medium or high risk aversion and they would expect a larger, than unconstrained investors, compensation for accepting the constrained allocation.
4. Historical Data with Extreme Values Exaggerated.

To check the robustness of the estimates of the proportionate opportunity cost, I include extremely high- and extremely low simulated asset returns in each data set of available assets (well-diversified portfolios and 9 (nine) asset portfolios).

The simulated extremely high- and extremely low asset returns are constructed in the following way. For the original data set for each historical time period I compute the average excess return across all assets in the data set. A historical time period with the highest average excess return across all assets defines the historical period with the highest returns. A historical time period with the lowest average excess return across all assets defines the historical period with lowest returns. Then, for the extreme historical periods only the deviation of each asset’s return from that asset’s intertemporal mean return is calculated. The deviations are doubled and then added back to assets’ intertemporal means. This way I create two fictional time periods with exaggerated high and exaggerated low returns. These fictional asset returns provide simulated extreme time periods to replace the time periods they were constructed from. The rest of the original data set remains unchanged.

Then I repeat the whole procedure for calculating the proportionate opportunity cost 1000 times in each case simulating two fictional data periods. This gives 1000 new $\theta$s from the 1000 data sets with extreme returns exaggerated. The results of this project are shown in Table 7 for the well-diversified number of assets and Table 8 for 9 (nine) assets.

Table 7
The proportionate opportunity cost, $(\theta - I)$, for various values of relative risk aversion with extreme returns exaggerated for the well-diversified number of assets
Table 8
The proportionate opportunity cost, \((\theta - 1)\), for various values of relative risk aversion with extreme returns exaggerated for nine assets

Out of all values considered for relative risk aversion, the lowest mean (over 1000 replications) of the proportionate opportunity cost for the well-diversified number of assets and for 9 (nine) assets corresponds to the high level of relative risk aversion of 31. The highest mean (over 1000 replications) of the proportionate opportunity cost for the well-diversified number of assets and 9 (nine) assets corresponds to the low level of risk aversion of 0.7.

The results of the second project confirm the results from the first one: as the level of relative risk aversion increases, the proportionate opportunity cost decreases, the better the constrained portfolio performs *ex ante* and the lower the proportion of initial wealth an investor required to stay constrained and accept equal portfolio shares and not optimal. Also, the same way as with the original data set, as the level of relative risk aversion increases, the standard deviation of the proportionate opportunity cost decreases. This means that as the level of risk aversion increases, as investors facing different asset sets become less risk tolerant, their perceptions of the optimal constrained portfolio strategy, even with extreme historical periods, are more similar to each other than perceptions of the strategy for investors with lower risk aversion.

The only difference between Tables 1 and 2 and Tables 7 and 8 is that the magnitude of the proportionate opportunity cost is bigger when extreme returns are exaggerated. These differences can be explained as follows. The calculation of the proportionate opportunity cost in Tables 7 and 8 was based on the historical asset returns with the extreme returns exaggerated. This exaggeration of historically occurred extreme
returns converted the original asset returns distribution, derived as shown in 4-9, into a distribution with “fatter” tails. This change in the distribution increases the probability for investors to end up with extremely low portfolio returns. Hence, the proportionate opportunity cost of using the constrained optimal portfolio instead of the unconstrained one in the case of exaggeration of extreme returns will be higher.

The lowest mean (over 1000 replications) of the proportionate opportunity cost for the well-diversified number of assets (Table 7) is 0.3% (0.003) and corresponds to the level of risk aversion of 31. This means that an investor with level of relative risk aversion of 31 being unconstrained will be equally happy as if he was constrained and had 0.3% more of initial wealth. The highest mean (over 1000 replications) of the proportionate opportunity cost, 17.6% (0.1676), corresponds to the very low level of relative risk aversion of 0.7. This means that an investor with the level of relative risk aversion of 0.7 being unconstrained will be equally happy as if he was constrained by had 17.6% more of initial wealth.

The lowest mean (over 1000 replications) of the proportionate opportunity cost for 9 (nine) assets (Table 8) is 1.1% (0.011) and corresponds to the level of risk aversion of 29 and higher. The highest mean (over 1000 replications) of the proportionate opportunity cost, 25.9% (0.259), corresponds to the very low level of relative risk aversion of 0.7.

V. Conclusion.

This paper investigated the opportunity cost incurred by investors when they use a sub-optimal portfolio strategy by placing identical number of dollars in \( n \) assets. CRRA
utility functions and the proportionate opportunity cost have been used. The opportunity cost has been calculated for different values of relative risk aversion.

The analysis has been done to find (a) the effect of restricting the number of assets in investors’ portfolios, and (b) the effect of restricting the initial wealth allocation between the assets or the use of “naïve” portfolio strategy.

My findings show that when the well-diversified number of assets is used the opportunity cost of using “naïve” portfolio strategy does not exceed 16.7% of initial wealth for the original historical asset return. When the number of available assets is restricted, in my case to nine (9) assets, the opportunity cost of investing sub-optimally increases to 20.4%. It shows that “naïve” portfolios are a very costly investment strategy to investors with low degree of risk aversion. As the degree of risk aversion increases the opportunity cost decreases for both the well-diversified number of assets and for nine (9) assets. It reaches 1.0% for the investors with degree of risk aversion of 29 and higher. And it is the same 1.0% for the well-diversified number of assets and for nine (9) assets.

For both data sets, for the original historical asset returns and for historical asset returns with extreme values exaggerated, as the level of relative risk aversion increases, the proportionate opportunity cost decreases.

An important conclusion from the analysis is that investors with low levels of risk tolerance will not benefit from full asset market participation, they will place a bigger portion of their initial wealth into Treasury bills whether they are using the “naïve” framework or not. Therefore, with such an investment strategy optimal unconstrained and “naïve” portfolios become almost identical with very small opportunity costs for the investors to incur should they decide to choose the later. Very small opportunity costs
suggest that constrained “naïve” portfolios are in fact investors’ optimal portfolio choice at high levels of risk aversion. And that is the reason why the conventional investment advise is to place an equal amount of dollars in all invested assets (Windcliff, Boyle (2004), Benartzi, Thaler (2001)).

Therefore, based on my calculations, I may conclude that for investors with very high levels of relative risk aversion (29 and above) “naïve” portfolios perform very well and show a fairly good approximation to the optimal portfolio strategy. Based on the popularity of such a strategy, I conclude that the majority of investors fall into the degree of risk aversion of 11 and above.
REFERENCES


Table 1

The Proportionate Opportunity Cost, \((\theta - 1)\), for Various Values of Relative Risk Aversion for the Well-Diversified Number of Assets

<table>
<thead>
<tr>
<th>Relative Risk Aversion, ((1-\gamma))</th>
<th>number of assets</th>
<th>Smallest</th>
<th>Mean</th>
<th>Largest</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>24</td>
<td>0.078</td>
<td>0.167</td>
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<td>0.061</td>
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<tr>
<td>0.999</td>
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<td>0.054</td>
</tr>
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<td>2</td>
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<tr>
<td>3</td>
<td>20</td>
<td>0.058</td>
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<td>0.297</td>
<td>0.042</td>
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<td><strong>Medium</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>0.004</td>
<td>0.014</td>
<td>0.093</td>
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<td>0.085</td>
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</tr>
<tr>
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<td>0.012</td>
<td>0.076</td>
<td>0.010</td>
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<tr>
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<td>0.004</td>
<td>0.011</td>
<td>0.068</td>
<td>0.009</td>
</tr>
<tr>
<td><strong>High</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>3</td>
<td>0.003</td>
<td>0.010</td>
<td>0.058</td>
<td>0.009</td>
</tr>
<tr>
<td>30</td>
<td>3</td>
<td>0.002</td>
<td>0.010</td>
<td>0.056</td>
<td>0.009</td>
</tr>
<tr>
<td>31</td>
<td>3</td>
<td>0.002</td>
<td>0.010</td>
<td>0.056</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Table 2

The Proportionate Opportunity Cost, \((\theta - 1)\), for Various Values of Relative Risk Aversion for 9 Assets

<table>
<thead>
<tr>
<th>Relative Risk Aversion, ((1-\gamma))</th>
<th>Smallest</th>
<th>Mean</th>
<th>Largest</th>
<th>Standard Deviation</th>
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</thead>
<tbody>
<tr>
<td><strong>Low</strong></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.094</td>
<td>0.204</td>
<td>0.434</td>
<td>0.057</td>
</tr>
<tr>
<td>0.999</td>
<td>0.089</td>
<td>0.131</td>
<td>0.217</td>
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<td>2</td>
<td>0.037</td>
<td>0.075</td>
<td>0.198</td>
<td>0.029</td>
</tr>
<tr>
<td>3</td>
<td>0.025</td>
<td>0.052</td>
<td>0.100</td>
<td>0.018</td>
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<td><strong>Medium</strong></td>
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<td></td>
</tr>
<tr>
<td>9</td>
<td>0.019</td>
<td>0.018</td>
<td>0.090</td>
<td>0.010</td>
</tr>
<tr>
<td>10</td>
<td>0.018</td>
<td>0.016</td>
<td>0.052</td>
<td>0.010</td>
</tr>
<tr>
<td>11</td>
<td>0.017</td>
<td>0.014</td>
<td>0.049</td>
<td>0.009</td>
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<tr>
<td>12</td>
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<td>0.012</td>
<td>0.038</td>
<td>0.009</td>
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<tr>
<td><strong>High</strong></td>
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<td></td>
</tr>
<tr>
<td>29</td>
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<td>0.010</td>
<td>0.018</td>
<td>0.006</td>
</tr>
<tr>
<td>30</td>
<td>0.003</td>
<td>0.010</td>
<td>0.018</td>
<td>0.005</td>
</tr>
<tr>
<td>31</td>
<td>0.003</td>
<td>0.010</td>
<td>0.017</td>
<td>0.005</td>
</tr>
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</table>
Table 3

Illustrative optimal portfolio shares for unconstrained portfolios and naïve (constrained) portfolios for different values of relative risk aversion for the well-diversified number of assets\(^1\)

<table>
<thead>
<tr>
<th># of An Asset</th>
<th>Relative Risk Aversion, ((1-\gamma)), equal to 0.7</th>
<th>Relative Risk Aversion, ((1-\gamma)), equal to 11</th>
<th>Relative Risk Aversion, ((1-\gamma)), equal to 31</th>
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<td></td>
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<td>Constrained</td>
<td>Unconstrained</td>
</tr>
<tr>
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<td>0.116</td>
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<tr>
<td>2</td>
<td>0.098</td>
<td>0.042</td>
<td>-0.403</td>
</tr>
<tr>
<td>3</td>
<td>0.57</td>
<td>0.042</td>
<td>0.924</td>
</tr>
<tr>
<td>4</td>
<td>-1.608</td>
<td>0.042</td>
<td>-0.607</td>
</tr>
<tr>
<td>5</td>
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<td>0.042</td>
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</tr>
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<tr>
<td>24(^2)</td>
<td>-9.606</td>
<td>0.042</td>
<td>-0.421</td>
</tr>
</tbody>
</table>

\(\text{E}(X^* R)\)^3

| Certainty Equivalent | 1.271 | 1.081 | 1.038 | 1.004 | 1.003 | 0.959 |

\(^1\) Numbers are not comparable across levels of risk aversion, because for each level of risk aversion a different set of available assets was used: a set giving an exact value of opportunity cost typical for that level of risk aversion.

\(^2\) The 24th asset is risk-free in nominal terms.

\(^3\) Monthly gross expected returns on portfolios.
Table 4

Illustrative optimal portfolio shares for unconstrained portfolios and naïve (constrained) portfolios for different values of relative risk aversion for nine assets

<table>
<thead>
<tr>
<th># of An Asset</th>
<th>Unconstrained</th>
<th>Constrained</th>
<th>Unconstrained</th>
<th>Constrained</th>
<th>Unconstrained</th>
<th>Constrained</th>
</tr>
</thead>
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<td>0.111</td>
<td>0.386</td>
<td>0.111</td>
<td>0.028</td>
<td>0.111</td>
</tr>
<tr>
<td>4</td>
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<td>-0.563</td>
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<td>0.001</td>
<td>0.111</td>
</tr>
<tr>
<td>5</td>
<td>2.174</td>
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<td>0.143</td>
<td>0.111</td>
<td>0.003</td>
<td>0.111</td>
</tr>
<tr>
<td>6</td>
<td>-2.604</td>
<td>0.111</td>
<td>0.673</td>
<td>0.111</td>
<td>0.004</td>
<td>0.111</td>
</tr>
<tr>
<td>7</td>
<td>3.075</td>
<td>0.111</td>
<td>0.199</td>
<td>0.111</td>
<td>0.048</td>
<td>0.111</td>
</tr>
<tr>
<td>8</td>
<td>-0.678</td>
<td>0.111</td>
<td>-0.245</td>
<td>0.111</td>
<td>0.056</td>
<td>0.111</td>
</tr>
<tr>
<td>9</td>
<td>-8.165</td>
<td>0.111</td>
<td>-0.661</td>
<td>0.111</td>
<td>0.856</td>
<td>0.111</td>
</tr>
<tr>
<td>E(X*′ R )³</td>
<td>1.324</td>
<td>1.108</td>
<td>1.067</td>
<td>1.009</td>
<td>1.001</td>
<td>0.969</td>
</tr>
<tr>
<td>Certainty Equivalent</td>
<td>1.112</td>
<td>1.039</td>
<td>1.011</td>
<td>0.997</td>
<td>0.997</td>
<td>0.952</td>
</tr>
</tbody>
</table>

1 Numbers are not comparable across levels of risk aversion because for each level of risk aversion a different set of available assets was used: a set giving an exact value of opportunity cost typical for that level of risk aversion.

2 The 9th asset is risk-free in nominal terms.

3 Monthly gross expected returns on portfolios.
Table 5

The ex post proportionate regret, \((\theta - 1)\), under the worst portfolio outcome for the well-diversified number of assets

<table>
<thead>
<tr>
<th>Relative Risk Aversion, ((1-\gamma))</th>
<th>Portfolios</th>
<th># of Assets</th>
<th>Smallest</th>
<th>Mean</th>
<th>Median</th>
<th>Largest</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>Unconstrained</td>
<td>24</td>
<td>3.974</td>
<td>9.919</td>
<td>9.231</td>
<td>21.797</td>
<td>3.935</td>
</tr>
<tr>
<td></td>
<td>Constrained</td>
<td>24</td>
<td>0.104</td>
<td>0.202</td>
<td>0.201</td>
<td>0.333</td>
<td>0.039</td>
</tr>
<tr>
<td>11</td>
<td>Unconstrained</td>
<td>7</td>
<td>0.024</td>
<td>0.079</td>
<td>0.070</td>
<td>0.584</td>
<td>0.053</td>
</tr>
<tr>
<td></td>
<td>Constrained</td>
<td>7</td>
<td>0.081</td>
<td>0.172</td>
<td>0.164</td>
<td>0.387</td>
<td>0.049</td>
</tr>
<tr>
<td>31</td>
<td>Unconstrained</td>
<td>3</td>
<td>0.005</td>
<td>0.022</td>
<td>0.016</td>
<td>0.113</td>
<td>0.017</td>
</tr>
<tr>
<td></td>
<td>Constrained</td>
<td>3</td>
<td>0.033</td>
<td>0.142</td>
<td>0.119</td>
<td>0.119</td>
<td>0.029</td>
</tr>
</tbody>
</table>

Table 6

The ex post proportionate regret, \((\theta - 1)\), under the worst portfolio outcome for nine assets

<table>
<thead>
<tr>
<th>Relative Risk Aversion, ((1-\gamma))</th>
<th>Portfolios</th>
<th>Smallest</th>
<th>Mean</th>
<th>Median</th>
<th>Largest</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>Unconstrained</td>
<td>0.974</td>
<td>9.206</td>
<td>4.653</td>
<td>41.880</td>
<td>5.417</td>
</tr>
<tr>
<td></td>
<td>Constrained</td>
<td>0.078</td>
<td>0.194</td>
<td>0.190</td>
<td>0.396</td>
<td>0.053</td>
</tr>
<tr>
<td>11</td>
<td>Unconstrained</td>
<td>0.033</td>
<td>0.077</td>
<td>0.074</td>
<td>0.278</td>
<td>0.025</td>
</tr>
<tr>
<td></td>
<td>Constrained</td>
<td>0.071</td>
<td>0.177</td>
<td>0.171</td>
<td>0.352</td>
<td>0.049</td>
</tr>
<tr>
<td>31</td>
<td>Unconstrained</td>
<td>0.009</td>
<td>0.027</td>
<td>0.026</td>
<td>0.141</td>
<td>0.011</td>
</tr>
<tr>
<td></td>
<td>Constrained</td>
<td>0.065</td>
<td>0.129</td>
<td>0.130</td>
<td>0.171</td>
<td>0.023</td>
</tr>
</tbody>
</table>
Table 7
The proportionate opportunity cost, \((\theta - 1)\), for various values of relative risk aversion with extreme returns exaggerated for the well-diversified number of assets

<table>
<thead>
<tr>
<th>Relative Risk Aversion, ((1-\gamma))</th>
<th># of Assets</th>
<th>Smallest</th>
<th>Mean</th>
<th>Median</th>
<th>Largest</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>25</td>
<td>0.061</td>
<td>0.176</td>
<td>0.167</td>
<td>0.724</td>
<td>0.096</td>
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<td>0.128</td>
<td>0.121</td>
<td>0.639</td>
<td>0.077</td>
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<tr>
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<td>0.075</td>
<td>0.068</td>
<td>0.535</td>
<td>0.054</td>
</tr>
<tr>
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<td>21</td>
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<td>0.073</td>
<td>0.057</td>
<td>0.312</td>
<td>0.041</td>
</tr>
<tr>
<td><strong>Medium</strong></td>
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</tr>
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<td>0.069</td>
<td>0.056</td>
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<td>0.037</td>
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</tr>
<tr>
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<td>0.035</td>
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<td>0.017</td>
</tr>
<tr>
<td>12</td>
<td>8</td>
<td>0.025</td>
<td>0.034</td>
<td>0.030</td>
<td>0.164</td>
<td>0.011</td>
</tr>
<tr>
<td><strong>High</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>4</td>
<td>0.009</td>
<td>0.004</td>
<td>0.002</td>
<td>0.011</td>
<td>0.009</td>
</tr>
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<td>3</td>
<td>0.005</td>
<td>0.003</td>
<td>0.002</td>
<td>0.009</td>
<td>0.009</td>
</tr>
<tr>
<td>31</td>
<td>3</td>
<td>0.005</td>
<td>0.003</td>
<td>0.002</td>
<td>0.009</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Table 8
The proportionate opportunity cost, \((\theta - 1)\), for various values of relative risk aversion with extreme returns exaggerated for nine assets

<table>
<thead>
<tr>
<th>Relative Risk Aversion, ((1-\gamma))</th>
<th>Smallest</th>
<th>Mean</th>
<th>Median</th>
<th>Largest</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.093</td>
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<td>0.649</td>
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<td>0.093</td>
<td>0.561</td>
<td>0.091</td>
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<tr>
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<tr>
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<td>0.007</td>
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<td>0.011</td>
<td>0.004</td>
<td>0.007</td>
<td>0.005</td>
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</tbody>
</table>