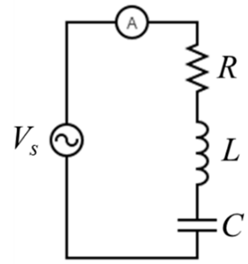
Q factor and bandwidth of RLC resonant circuit

The Q-factor of a system is defined as $Q = \dfrac{2\pi \times \text{stored energy}}{\text{dissipated energy per cycle}}$ (at resonance).

We now have the RLC resonant circuit as



The current amplitude is

$$I = \frac{V_s}{Z_{tot}} = \frac{V_s}{R + j\left(\omega L - \dfrac{1}{\omega C}\right)} \Rightarrow |I| = \frac{|V_s|}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} = \frac{|V_s|/R}{\sqrt{1 + \left(\omega L - \dfrac{1}{\omega C}\right)^2 \Big/ R^2}}.$$

Let the current be $I(t) = |I| e^{i\omega t}$, i.e., its phase is 0.

The voltage on the capacitor is then $V_C(t) = IX_C = \dfrac{|I|}{j\omega C} e^{i\omega t} = \dfrac{|I|}{\omega C} e^{i\left(\omega t - \frac{\pi}{2}\right)}$.

The energy stored in the inductance is $U_L = \dfrac{1}{2} L \left(|I| \cos \omega t\right)^2$.

The energy stored in the capacitor is $U_C = \dfrac{1}{2} C \left[|V_c| \cos\left(\omega t - \dfrac{\pi}{2}\right)\right]^2 = \dfrac{1}{2}\dfrac{1}{\omega^2 C}\left(|I| \sin \omega t\right)^2$.

At resonance, $\omega_0 = \dfrac{1}{\sqrt{LC}}$, we have

$$U_L + U_C = \frac{1}{2} L \left(|I| \cos \omega t\right)^2 + \frac{1}{2}\frac{1}{\omega^2 C}\left(|I| \sin \omega t\right)^2 = \frac{1}{2} L |I|^2.$$

The energy dissipated by the resistor per cycle is

$$U_R = \int_0^T \left(|I|\cos\omega_0 t\right)^2 R dt = \frac{1}{2}|I|^2 RT = \frac{1}{2}|I|^2 R \times \frac{2\pi}{\omega_0}.$$

After these preparations, the Q factor is now

$$Q = \frac{2\pi \times \text{stored energy}}{\text{dissipated energy per cycle}} = \frac{2\pi \times (U_L + U_C)}{U_R}$$

$$= \frac{2\pi \times \frac{1}{2} L |I|^2}{\frac{1}{2}|I|^2 R \times \frac{2\pi}{\omega_0}} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{|X_L|}{R} = \frac{|X_C|}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

The bandwidth is given by $\Delta\omega = \omega_2 - \omega_1$, where ω_1 and ω_2 satisfy $\frac{|I(\omega_{1,2})|^2}{|I|^2_{\max}} = \frac{1}{2}$.

From $|I| = \frac{|V_s|/R}{\sqrt{1 + \left(\omega L - \frac{1}{\omega C}\right)^2 \Big/ R^2}}$ above, this means $\omega L - \frac{1}{\omega C} = \pm R$.

The solution is easily found to be $\omega_{2,1} = \frac{\pm RC + \sqrt{(RC)^2 + 4LC}}{2LC}$.

Therefore, $\Delta\omega = \omega_2 - \omega_1 = \frac{R}{L}$. When measured by frequency, we have the bandwidth

$\Delta f = \frac{1}{2\pi}\frac{R}{L} = \frac{f_0}{Q}$, and the famous relation of $Q = \frac{f_0}{\Delta f}$.

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