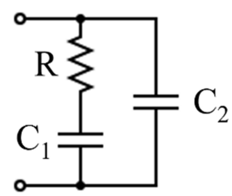
Failure of arctan

To find the phase of a complex number, most textbooks tell us

$$\arg(x + jy) = \arctan\frac{y}{x}. \quad (1)$$

At the same time, we know that the range of the phase of a complex number is $-\pi < \arg(x + jy) \le \pi$, while the range of the arctan function, when specially defined, is $-\pi / 2 \le \arctan(z) \le \pi / 2$. Therefore, the above equation cannot be always true even in principle.

An example is shown here. Let us find the phase of the impedance of the following combination:



The total impedance is

$$Z = \frac{\left(R + \frac{1}{j\omega C_1}\right)\frac{1}{j\omega C_2}}{\left(R + \frac{1}{j\omega C_1}\right) + \frac{1}{j\omega C_2}} = \frac{1 + j\omega RC_1}{-\omega^2 RC_1C_2 + j\omega(C_1 + C_2)}.$$

If we now naively follow the prescription of Eq. 1, we will get

$$\arg(Z) = \arctan(\omega RC_1) - \arctan\frac{\omega(C_1 + C_2)}{-\omega^2 RC_1C_2} = \arctan(\omega RC_1) + \arctan\frac{\omega(C_1 + C_2)}{\omega^2 RC_1C_2}. \quad (2)$$

This cannot be correct, because the denominator $-\omega^2 RC_1C_2 + j\omega(C_1 + C_2)$ has a phase between $\pi/2$ and π (since $x < 0$ and $y > 0$), which is beyond the range of arctan (y/x). To correct this problem, we notice

$$\arg(-\omega^2 RC_1C_2 + j\omega(C_1 + C_2)) = \pi - \arctan\frac{\omega(C_1 + C_2)}{\omega^2 RC_1C_2},$$

so the correct answer for the phase of Z is

$$\arg(Z) = \arctan(\omega RC_1) - \left[\pi - \arctan\frac{\omega(C_1+C_2)}{\omega^2 RC_1C_2}\right] = \arctan(\omega RC_1) + \arctan\frac{\omega(C_1+C_2)}{\omega^2 RC_1C_2} - \pi$$

The naive Eq. 2 has missed an angle of π, which may lead to incorrect physics.

In summary, when using $\arg(x+jy) = \arctan\frac{y}{x}$ to find the phase of a complex number, make sure we have $x > 0$. If $x < 0$, the phase is then

$$\arg(x+jy) = \arctan\left[(-1)\times(-x-jy)\right] = \text{sign}(y)\times\pi + \arctan\frac{y}{x}.$$

Here $\text{sign}(y) = \frac{y}{|y|}$ is used so that the resulted angle always satisfies $-\pi < \arg(x+jy) \le \pi$.

Pengqian Wang

September 26, 2025