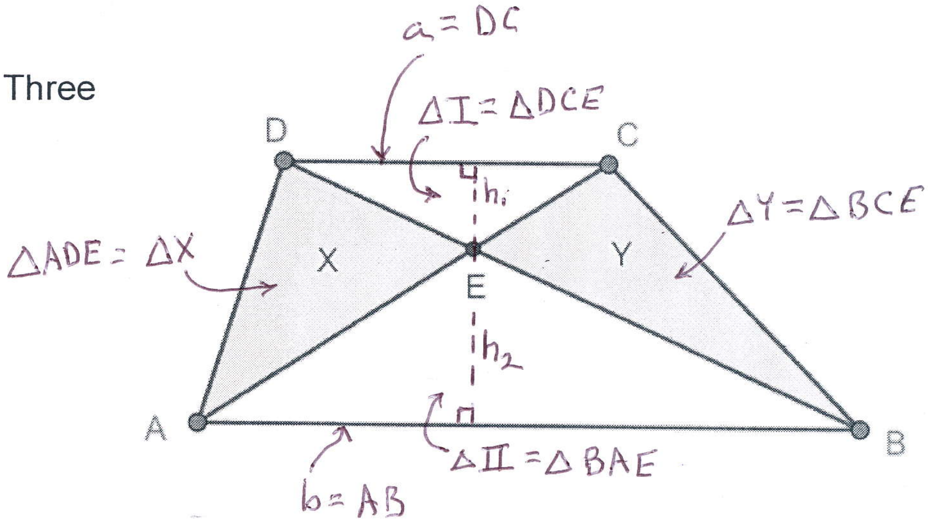
Triangles in a Trapezoid - Three

ABCD is a trapezoid.

Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point E.



Prove:
$Area(\Delta ADE)$ is the geometric mean of $Area(\Delta DCE)$ and $Area(\Delta BAE)$

Note: $Area(\Delta ADE) = Area(\Delta BCE)$, so $Area(\Delta BCE)$ is also the geometric mean of areas of the top and bottom triangles.

Proof:

We will refer to the Δs as ΔI, ΔII, ΔX, and ΔY.

Let h_1 and h_2 be the heights of ΔI and ΔII, respectively.

Compute areas:

$$A_X = A_Y = \frac{1}{2}b(h_1+h_2) - \frac{1}{2}bh_2$$

$$A_X = A_Y = \frac{1}{2}bh_1$$

$$A_I = \frac{1}{2}ah_1 \qquad A_{II} = \frac{1}{2}bh_2 \quad (*)$$

$\Delta DCE \sim \Delta BAE$ $(\Delta I \sim \Delta II)$ by A.A. (since we have parallel lines and alternate interior angles $\cong$.)

The heights are in the same ratio as the sides.

$$\frac{a}{h_1} = \frac{b}{h_2} \Rightarrow h_2 = \frac{b}{a}h_1$$

From $(*)$ $A_{II} = \frac{1}{2}b\left(\frac{b}{a}h_1\right) = \frac{1}{2}\frac{b^2}{a}h_1$

$$A_I \cdot A_{II} = \frac{1}{2}ah_1 \cdot \frac{1}{2}\frac{b^2}{a}h_1 = \frac{1}{4}\cdot b^2 \cdot h_1^2$$

$$\sqrt{A_I \cdot A_{II}} = \frac{1}{2}bh_1 \quad \leftarrow \text{same as } A_X + A_Y$$

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$$A_X = A_Y = \sqrt{A_I \cdot A_{II}}$$