Three Pythagorean Means: Timeless, Central, and of Enduring Value to This Day

Overview of the Harmonic, Geometric, and Arithmetic Means

First Look – The Idea

Arithmetic Mean – *the additive average.* Add the numbers and divide by 2. (Dividing by 2 is 'un-adding.')

Geometric Mean – *the multiplicative average.* Multiply the numbers and take the square root. (Square root is 'un-multiplying.')

Harmonic Mean – *the reciprocal of the average of the reciprocals.* That is, take reciprocals, then average (add and divide by 2), then take the reciprocal. (Taking the reciprocal is 'un-reciprocal.')

Formulas – means of two numbers a and b

Arithmetic Mean $= \frac{a+b}{2}$

Geometric Mean $= \sqrt{ab}$

Harmonic Mean $= \frac{2ab}{a+b}$

Wonderful Relationship

If $a \leq b$, then

$$a \leq \frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq b$$

(With equality occurring when $a = b$.) ☺

Example

For $a = 4$ and $b = 9$:

Arithmetic Mean $= \frac{4+9}{2} = \frac{13}{2} = 6\frac{1}{2}$

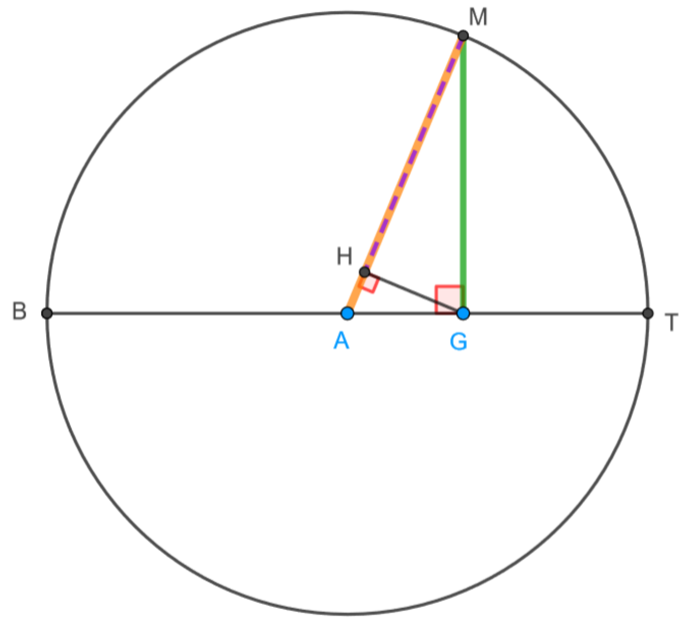
Geometric Mean $= \sqrt{4 \cdot 9} = \sqrt{36} = 6$

Harmonic Mean $= \frac{2 \cdot 4 \cdot 9}{4+9} = \frac{72}{13} = 5\frac{7}{13}$

(We can also take the means of a list of many numbers.)

See discussion of applications on the next page.

Circle A with diameter BT



Discussion

Adopted from http://mathforum.org/ – When asked when to apply the arithmetic, geometric, and harmonic means for a certain data set, *Ask Dr. Math* (of The Math Forum) responded.

The choice depends on the meaning of the numbers and specifically how the numbers naturally combine.

The basic idea is that a **mean** is a number that can be used in place of each number in a set, for which the *net effect* will be the same as that of the original set of numbers. What determines which mean to use is the way in which the numbers act together to produce that net effect.

For example, if you are looking for a mean amount of rainfall, you note that the total amount of rain, which affects crop growth, etc., is found by ADDING the daily numbers; so if you add them up and divide by the number of days, the resulting ARITHMETIC mean is the amount of rain you could have had on EACH of those days, to get the same total.

If you have several successive price markups, say by 5% and then by 6%, and want to know the mean markup, you note that the net effect is to first MULTIPLY by 1.05 and then by 1.06, equivalent to a single markup of 1.05*1.06 = 1.113; taking the square root of this, if you had TWO markups of 5.499% each, you would get the same result. This is the GEOMETRIC mean. In general, you use it where the product is an appropriate "total"; another example is when you combine several enlargements of a picture.

If you want the mean speed of a car for a round trip (using different speeds to and from) you want the HARMONIC mean. For example, if you drive 60 mph from Start City to Springfield and drive 40 mph from Springfield back to Start City (due to fog perhaps), then the average speed for the round trip is 48 mph. When one drives the same distance (not time) at each of several speeds, then the net effect of all the driving (the total time taken) is found by dividing the common distance by each speed to get the time for that leg of the trip and then adding up those times. The constant speed that would take the same total time for the whole trip is the HARMONIC mean of the speeds. This amounts to the reciprocal of the arithmetic mean of the RECIPROCALS of the individual speeds. In general, we use the harmonic mean when the numbers naturally combine via their reciprocals. Another example is combining resistances in a parallel electrical circuit.

(Statisticians also use other means, such as the quadratic mean.)