More about the falling raindrop

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A simple strategy is presented for solving the “inverse rocket” problem of a particle accumulating material from a medium through which it falls vertically. Some forms of drag can also be easily included, thereby changing the constant acceleration to a more realistic value.

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Sokal has introduced a general version of the problem of a raindrop accumulating mass as it falls through mist and solved it using the chain rule and an integrating factor. In this Note, it is shown that his solution can be simplified by using momentum as the dependent variable, making the solution more accessible to introductory physics majors. In addition, the problem is further generalized to include a drag force.

In the absence of drag, Newton’s second law for the drop is

\[ \frac{dp}{dt} = mg, \]

where \( p=mv \) is its momentum and the gravitational field \( g \) is assumed to be constant. The mass accretion rate is assumed to scale with powers of the mass (or size) and speed of the drop,

\[ \frac{dm}{dt} = \lambda m^\alpha \nu^\beta = \lambda m^{\alpha-\beta} \nu^\beta, \]

with values of the parameters \( \lambda > 0, \ 1 > \alpha \geq 0, \) and \( \beta \geq 0 \) chosen to ensure stability. Dividing Eq. (1) by Eq. (2) gives

\[ p^\beta dp = \frac{g}{\lambda} m^{1+\beta-\alpha} dm, \]

which has the general solution

\[ \frac{p^{1+\beta}}{1+\beta} = \frac{gm^{2+\beta-\alpha}}{\lambda(2+\beta-\alpha)} + C \frac{1}{1+\beta}, \]

where the last term is a constant of integration. If we substitute \( p=mv \), Eq. (4) becomes the same as Sokal’s solution.

Drag can be included by adding a term to the right-hand side of Eq. (1),

\[ \frac{dp}{dt} = mg - \epsilon m^\alpha \nu^\gamma = mg - \epsilon m^{\alpha-\gamma} p^\gamma, \]

with \( \epsilon \geq 0 \) and \( \gamma > 0 \) and where the same exponent \( \alpha \) for \( m \) is used for the drag and mass accretion terms because both effects are expected to scale similarly with the size of the drop.\(^2\) Solving Eqs. (2) and (5) simultaneously is straightforward when \( \gamma = 1 + \beta \), which includes the important special cases \( \gamma = 1 \) and \( \beta = 0 \) (linear drag with speed-independent mass accretion) and \( \gamma = 2 \) and \( \beta = 1 \) (quadratic drag with linear speed accretion). The solution when \( \gamma = 1 + \beta \) can be obtained by dividing Eq. (5) by Eq. (2), which leads to

\[ \frac{dp}{dm} + \frac{\epsilon}{\lambda} m^{1+\beta-\alpha} p^{-\beta} = \frac{gm^{1+\beta-\alpha}}{\lambda}. \]

The left-hand side of Eq. (6) suggests changing the dependent variable to \( u = m^{(1+\beta)/\alpha} \), which yields

\[ u^{1+\beta} du = \frac{g}{\lambda} m^{1+\beta-\alpha} dm \]

as a generalization of Eq. (3). This separated equation can be integrated. Rewriting the resulting solution in terms of \( v \) gives

\[ v^{1+\beta} = \frac{g(1+\beta)m^{1-\alpha}}{\lambda(2+\beta-\alpha) + \epsilon(1+\beta)} + \frac{C}{m^{1+\beta}(1+\epsilon/\lambda)}. \]

We let \( C = 0 \) (assuming \( v = 0 \) when \( m = 0 \)) in Eq. (8), differentiate it with respect to time \( t \), and then substitute Eq. (2) into the right-hand side and find the acceleration \( a = dv/dt \) to be a constant,

\[ a = \frac{ng}{1+\epsilon(1-n)/\lambda}, \]

where \( n = (1-\alpha)/(2+\beta-\alpha) \). This acceleration reduces to Sokal’s drag-free result for \( \epsilon = 0 \). By including a value for \( \epsilon \) of the order of 100\( \lambda \), the acceleration of a raindrop can be reduced to a more realistic value\(^4\) of a few thousandths of \( g \) (for typical values of \( n \) of a few tenths).

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\(^3\)A similar observation applied to Eq. (8) divided by \( v^\beta \) in Ref. 1 suggests a change of dependent variable to \( p=mv \).


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The journal title of Ref. 21 should read Appl. Phys. B (not Opt. Photonics News). 1 We are grateful to Y. Yang for pointing out the error.

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Kholmetskii, Missevitch, and Yarman 1 recently reanalyzed the usual classical derivation of spin-orbit coupling in hydrogenlike atoms and found a result “in qualitative agreement with the solution of the Dirac–Coulomb equation for hydrogenlike atoms.” However, their result is based on an equation of translational motion of the electron 2 that omits any contribution due to the existence of the “hidden” momentum of the electron intrinsic magnetic dipole moment in the electric field of the nucleus. Accounting for hidden momentum is necessary to obtain conservation of linear momentum in the interaction of a classical current-loop magnetic dipole with a point charge. 3–5 Classical electrodynamics textbooks 6,7 recognize this need. It also has been argued 8 that the hidden momentum of the electron intrinsic magnetic moment must be incorporated in the laboratory-frame analysis of atomic spin-orbit coupling to obtain an equation of motion of the electron polarization that is consistent between the laboratory frame and the electron rest frame.

The observation that the spin-orbit coupling magnitude must involve the magnitude of the binding force is correct, 1 but including hidden momentum in the electron equation of translational motion has the effect of approximately halving the non-Coulomb force on the electron compared to its value obtained by omitting the hidden momentum. In contrast and apart from the issue of whether hidden momentum is associated with intrinsic as well as classical current-loop magnetic moments, if hidden momentum is omitted from the analysis, then the force on the nucleus due to the electron differs from the force on the electron due to the nucleus. Thus, if the hidden momentum contribution is omitted, the binding energy including the spin-orbit coupling cannot be consistently calculated. Furthermore, because the spin-orbit coupling magnitude calculated in Sec. III of Ref. 1 is based on the non-Coulomb force acting on the electron, it will be halved when the hidden momentum is incorporated into the analysis, and the resulting spin-orbit coupling value will be in disagreement with the experiment.

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2 Reference 1, Eq. (26).


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We thank D. C. Lush for his comment on our paper, pointing out the known role of hidden momentum in the classical understanding of spin-orbit coupling in hydrogenlike atoms. In our approach to this problem, we followed the historical analysis of spin-orbit interaction that started with the original work by Thomas and Frenkel. Our main goal was to consider the interaction between the electric field of the nucleus and electric dipole moment on the orbiting electron, which is usually missed in the laboratory frame analysis. Another essential point in Ref. 2 is the detailed application of the second Bohr postulate, in which we use the momentum of the electron defined by its equation of motion with the inclusion of spin effects. In such an approach, the hidden momentum, which is included in the equation of motion of the electron, is also accounted for by the second Bohr postulate. The force associated with the time variation of the hidden momentum was not considered for by the second Bohr postulate. Hence, in our approach, it leaves the expression for the spin-orbit coupling unmodified. In the following, we will show this result.

We agree with Lush that the rigorous classical analysis of spin-orbit coupling must also include the force due to the time variation of the hidden momentum. Additionally, as shown in Ref. 5 and discussed in Ref. 6, the expression for the force acting on a particle with a magnetic moment derived from Dirac’s equation in the classical limit already includes the contribution due to the time variation of the hidden momentum:

\[ F_{\text{hidden}} = -\frac{d}{dt}\frac{\mu \times E}{c}, \]  

(1)

given in the rest frame of the particle. (We follow the notation in Ref. 2.) The force term in Eq. (1) should be included in the equation of motion of the electron in the laboratory frame [Ref. 2, Eq. (26)], which thus becomes

\[ \frac{d}{dt}(mv) = -\frac{Ze^2r}{r^3} + (p \cdot \nabla)\frac{Ze}{r^2} - F'_{\text{hidden}}, \]  

(2)

where \( F'_{\text{hidden}} \) is defined in the laboratory frame (the rest frame of an infinitely massive nucleus).

For simplicity, we again assume as in Ref. 2 that the electron’s spin \( s \) is orthogonal to the rotation plane and its magnetic moment does not vary with time. In addition, we do the calculations to order \( (v/c)^2 \) and set \( F_{\text{hidden}} = F'_{\text{hidden}} \). With these limitations, the equation of motion becomes

\[ \frac{d}{dt}(mv) = -\frac{Ze^2r}{r^3} + (p \cdot \nabla)\frac{Ze}{r^2} - \frac{d}{dt}\frac{\mu \times E}{c} = \frac{\mu \times E}{c}, \]  

(3)

[compare with Ref. 2, Eq. (27)]. For the circular motion of the electron around the nucleus, the electric field at the location of the electron remains constant in magnitude and varies in direction so that

\[ \frac{dE}{dt} = \omega \times E, \]  

(4)

where \( \omega \) is the angular rotation frequency of the electron. The latter equality lets us transform the last term on the right-hand side of Eq. (3) to

\[ -\mu \times E/c dt = -\mu \times (\omega \times E)/c = (\mu \cdot \omega)E/c = \frac{pZe}{r^4}. \]  

(5)

Here we used the vector identity \( a \times (b \times c) = b(a \cdot c) - c(a \cdot b) \), taken into account that the vector \( E \) is orthogonal to \( \mu \), and the vector \( \mu \) is collinear with \( \omega \), and assumed \( \mu = pe/cv \) and \( v = wr \). Thus, we see that the force acting on the electron due to the variation of its hidden momentum [the third term on the right-hand side of Eq. (3)] is half of the force acting on the electron due to its relativistic polarization (the second term on right-hand side of Eq. (3) with the opposite sign). This result is in agreement with the statement by Lush “…including hidden momentum in the electron equation of translational motion has the effect of approximately halving…the non-Coulomb force…” Hence, Eq. (3) reads

\[ \frac{d}{dt}(mv) = -\frac{Ze^2r}{r^3} - \frac{pZe}{r^4}. \]  

(6)

As in Ref. 2, we use the second Bohr postulate with the electron’s momentum computed from Eq. (6) and again obtain the equality

\[ \Delta U(p) = 0, \]  

(7)

where \( \Delta U \) is defined via Eq. (24) in Ref. 2, which yields the usual expression for the spin-orbit coupling. However, Eq. (36) in Ref. 2 for the radii of the electron’s suborbits
acquires the factor 1/2 in the spin-dependent term,
\[ r_\pm = n^2 r_B \left( 1 \pm \frac{s (Z\alpha)^2}{2n^4} \right). \]  
(8)

The derivation of Eqs. (7) and (8) can be generalized to an arbitrary spatial orientation of \( s \). Thus, the inclusion of the hidden momentum in the equation of motion of the classical electron in hydrogen-like atom does not influence the spin-orbit splitting [see Ref. 2, Eq. (23)]. This conclusion supports the result obtained in Ref. 7.

The difference in the numerical coefficient in Eq. (8) in comparison with the result in Ref. 2, Eq. (36), is insignificant because the classical estimate of the average orbital radius of electron does not take into account, for example, its dependence on orbital quantum number and thus agrees with the exact quantum solutions only qualitatively.

Finally, we comment on the statement by Lush “...if hidden momentum is omitted from the analysis, the force on the nucleus due to the electron will differ from the force on the electron due to the nucleus,” which is undoubtedly correct in the classical approach. However, we note that both the velocity-dependent (bound) and acceleration-dependent (radiation) field component equally participate in maintaining the momentum balance in the classical system of orbiting electron plus resting nucleus, whereas quantum mechanically, the bound electron does not radiate in stationary energy states. Thus, the results of classical electrodynamics usually cannot be extended to the bound wavelike particles, although the hidden momentum must also participate in the momentum conservation law for purely bound quantum systems. However, such an extension requires further analysis (see, for example, Ref. 8).

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