

THE COST OF A SHORT-SELLING CONSTRAINT – WELFARE IMPLICATIONS FOR INVESTORS UNDER UNCERTAINTY

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Abstract

In this paper I analyze investors' welfare losses from being restricted from short selling. To measure those losses I use the concept of the proportionate opportunity cost along with various CRRA utility functions. Two sets of asset returns are used with a VAR in generating joint returns distributions: the original historical asset returns data set, and the historical asset returns with extreme values exaggerated. In each case 1,000 alternative sets of assets including one with a risk-free nominal return are randomly made available for investment. I show that the optimal portfolio strategy with the short-selling constraint performs almost as well as the unconstrained portfolio strategy for investors with medium levels of risk aversion, and performs as well as the unconstrained portfolio strategy for investors with high levels of risk aversion. The results, derived from the original historical asset returns data set show that investors' welfare losses reach 12.8% of initial wealth when risk aversion is low. With extreme returns exaggerated investors' welfare losses reach 13.5% of initial wealth. The results in both cases indicate that less risk-averse investors experience greater welfare losses, and that the short-selling constraint reduces the cost of sub-optimal diversification.

Keywords: Probability distribution function of stock returns; Proportionate opportunity cost; Optimal portfolio strategy; Investors' welfare losses

Running Title: The Cost of a Short-Selling Constraint

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1. Introduction

Short-selling is a legitimate trading strategy. By short-selling an investor will receive extra money to invest and will make positive returns if the shorted assets' price rises by less than that of the assets in which the extra funds are placed. At the same time it is a risky strategy because the investor will lose money if the price of the shorted asset rises by more than that of the assets in which the extra funds are placed. An optimal portfolio may require an investor to hold extremely long or extremely short investment positions. These extreme long or short holding positions sometimes are difficult to implement in practice because investors face constraints on their portfolio holdings. For example, Regulation T, which applies to almost all investors, institutional as well as individual, requires 50% margin (an investor may not use more than 50% of his initial wealth for short-selling).

Any short-selling restriction will reduce investment opportunities for an investor, and, therefore, will create a constrained portfolio strategy.

Economists researching household portfolio choices find that asset allocation decisions are influenced by several factors including degree of risk aversion, education level (which affects the investor's ability to gather and process financial information), level of initial wealth, age category, borrowing constraints, family size, and gender: Bertaut (1998), Bertaut and Haliassos (1997), Guiso, Jappelli, and Terlizzese (1996), Guiso, Haliassos, and Jappelli (2002), Haliassos and Michaelides (2003), Jagannathan and Kocherlakota (1996), Jianakoplos and Bernasek (1998), Kennickell, Starr-McCluer, and Surette (2000), Uhler and Cragg (1971), Ameriks and Zeldes (2000), Attanasio and Hoynes (2000).

Although these studies provide a comprehensive catalogue of factors that affect household portfolio choice, they do not address the impact of sub-optimal asset allocation on household-investors' utility when investors are restricted from short-selling. Tepla (2000) used mean-variance portfolios and worked in a dynamic environment. She found that short-sale restrictions create inefficient portfolios and the opportunity cost in her case is the reduced number of assets to invest in which might hurt the diversification aspect.

Wang (1998) dealt with the mean-variance framework as well, and worked with one-period model and with two types of short-selling constraints: (1) no short position of an asset is allowed to be greater than 50% of initial wealth; and (2) all short positions are prohibited. Wang findings are: short-sale restrictions do create inefficient portfolios and the opportunity cost is the loss in expected portfolio return. And as a short-selling constraint becomes more restrictive the loss in the expected portfolio return increases. Best and Grauer (1991) also worked with the mean-variance framework. The question they asked was: how would a change in the mean return of a security in an investor's portfolio affect the portfolio return. They found out also that the effect of a short-selling constraint in their case was a reduced number of assets chosen to invest in.

Luttmer (1996) pointed out that short-selling constraints on Treasury bills prevent investors from exploiting the equity premium by borrowing at the Treasury bill rate and investing in the stock market. So, the cost of a short-selling constraint in his case is the lower expected portfolio return.

Although these research works may explain the effect of different short-selling restrictions on portfolio performance, the question that remains unanswered is: How big is one's welfare loss if one restricts oneself from short-selling?

Estimating utility losses for an investor who restricts himself from short-selling is important for analyzing portfolio allocation decisions.

The best way to approach the problem of calculating the opportunity cost of short-selling restrictions is to use optimal portfolios (not mean-variance efficient as Tepla did, but rather globally optimal) that can be found through an optimization procedure.

The procedure followed in this paper for calculating this proportionate opportunity cost includes random asset selection for investors' portfolios, estimation of a vector autoregressive process, derivation of the joint probability distribution function of asset returns, and computing optimal constrained and unconstrained portfolios.

In this paper I show that with a nominally risk-free asset the optimal portfolio strategy with the short-selling constraint performs almost as well as the unconstrained portfolio strategy for investors with medium levels of risk aversion, and performs as well as the unconstrained portfolio strategy for investors with high levels of risk aversion. The results, derived from the original historical asset returns data set with no exaggeration of extreme returns, show that investors' welfare losses reach 12.8% of initial wealth for investors with low risk aversion; and become even larger when extreme returns in the original historical data set are exaggerated.

The second section of this paper introduced the proportionate opportunity cost; the third section describes the procedure of random asset selection for investors' portfolios, of inferring the joint probability distribution function of asset returns, of computing constrained and unconstrained optimal portfolios, and of calculating the proportionate opportunity cost. The fourth section discusses the results of the study, and the fifth section concludes.

II. Proportionate opportunity cost.

In order to measure welfare losses from investing in constrained portfolios, I will compare expected utility from the optimal portfolio with a short-selling restriction, with that from the optimal unconstrained portfolio with no short-selling restrictions, by using the concept of opportunity cost as developed by Brennan and Torous (1999) and Tew, Reid and Witt (1991).

The proportionate opportunity cost is the best way to measure investors' welfare losses when they face any kind of constraint on their holdings. Under the assumption of the constant relative risk aversion utility function

$$(1) \quad U(\tilde{w}) = \begin{cases} \frac{1}{\gamma} \tilde{w}^\gamma, & \gamma < 1, \gamma \neq 0, \tilde{w} > 0 \\ -\infty, & \tilde{w} \leq 0 \end{cases}$$

the proportionate opportunity cost (willingness to accept payment as compensation for being constrained) can be calculated as $\theta - 1.0$ where θ is defined by

$$(2) \quad EU(\theta w_0 \tilde{R}^c) = EU(w_0 \tilde{R}^u)$$

where w_0 is the initial wealth, and \tilde{R}^u and \tilde{R}^c are the stochastic returns per dollar invested in the unconstrained and constrained portfolios. Solving (2) gives

$$(3) \quad \theta = \left[\frac{E(\tilde{R}^\gamma)^{unconstrained}}{E(\tilde{R}^\gamma)^{constrained}} \right]^{\frac{1}{\gamma}}.$$

Under CRRA utility function θ also equals the ratio of certainty equivalents of the unconstrained optimal portfolios and constrained optimal portfolios (with short-selling restrictions). Since the ratio of certainty equivalents is unitless and in particular has no time units, the proportionate opportunity cost, $\theta - 1.0$, is timeless. But its numerical value

depends on a number of months until horizon, i.e. with the investment horizon of T months the proportionate willingness to accept payment is θ^T .

III. The Procedure

A. Portfolio Formation

The data set used is monthly historically occurring asset returns over the ten-year period from January 1994 through December 2003. With different time units, the conclusions might be affected. But with quarterly asset returns I would need to extend the time period to 1974.I to 2003.IV, and with annual asset returns the new time period will be from 1884 to 2003 just to get the same 120 data points. In both cases I would have to deal with very old asset returns that might not accurately reflect the true probability distribution facing current investors. The choice of monthly time units is also consistent with the studies of Simaan (1993) and Kroll, Levy, and Markowitz (1984).

B. Asset Selection

The procedure of calculating the proportionate opportunity cost for different levels of risk aversion will be performed 1,000 times in each case using a well-diversified optimal portfolio with $n-1$ randomly picked nominally risky assets and Treasury bills as the nominally risk-free asset. The well-diversified number of assets, n , for an optimal portfolio for every level of risk aversion is taken from A. A. Melkumian and A. V. Melkumian (2009)

Alternatively, the procedure of calculating the proportionate opportunity cost of the short-selling constraint for different levels of risk aversion will be repeated 1,000

times for ten assets as well: nine randomly picked nominally risky assets and Treasury bills. The ten assets are suggested by the literature on constrained portfolio strategies, e.g. Tew, Reid and Witt (1991) and is used to see how the proportionate opportunity cost of a short-selling restriction changes with the change in the level of risk aversion only.

The first step in the procedure of calculating the proportionate opportunity cost of a short-selling restriction is to pick at random $n-1$ nominally risky assets, as the well-diversified number of nominally risky assets. Next step is to get expected values of real returns for the n assets for time $T+1$: for the $n-1$ nominally risky assets and for nominally risk-free Treasury bills. In real terms, though, there is no risk-free asset. Returns on Treasury bills are risk-free only in nominal terms. But in time-series data inflation will be uncertain in any period and, thus, so will the real rate of return on Treasury bills. Therefore, the n assets in real terms will all be risky assets. The same procedures are also conducted with ten assets instead of n .

C. Vector Autoregressions of Returns

To get expected values of real returns for the case of n assets at time $T+1$, the portfolio formation period, I estimate a vector autoregressive process (VAR). The next steps are to derive the joint probability distribution for the n assets' real returns, and, finally, to construct optimal constrained and optimal unconstrained portfolios.

To derive the joint probability distribution of empirical deviations from the VAR-estimated conditional means for those randomly picked asset returns the following procedure is used.

The nominal return on asset i at time t minus the nominal return on Treasury bills at time t gives the excess return on asset i ($x_{i,t}$) at time t for $i=1, \dots, n-1$ and for $t=1, \dots, T$.

Running a VAR for excess returns of those $n-1$ assets and realized inflation, as

$$(4) \quad \begin{bmatrix} x_{1,t} \\ \vdots \\ x_{n-1,t} \\ \pi_t \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} + \begin{bmatrix} v_{1,1}(L) & \cdot & \cdot & v_{1,n}(L) \\ \cdot & \cdot & \cdot & \cdot \\ v_{n-1,1}(L) & \cdot & \cdot & v_{n-1,n}(L) \\ v_{n,1}(L) & \cdot & \cdot & v_{n,n}(L) \end{bmatrix} \begin{bmatrix} x_{1,t} \\ \vdots \\ x_{n-1,t} \\ \pi_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \vdots \\ \varepsilon_{n-1,t} \\ \varepsilon_{\pi,t} \end{bmatrix},$$

$\{\hat{c}_i\}$, $\{\hat{\varepsilon}_{i,t}\}$ and $\{\hat{v}_{i,k}(L)\}$ are obtained, where

$$(5) \quad \hat{v}_{i,k}(L) = \hat{\delta}_{i,k}^1 L^1 + \hat{\delta}_{i,k}^2 L^2 + \dots$$

The vector of conditional expected values of excess returns for time $T+1$ and expected inflation for time $T+1$ is calculated as:

$$(6) \quad \begin{bmatrix} E_T x_{1,T+1} \\ \vdots \\ E_T x_{n-1,T+1} \\ E_T \pi_{T+1} \end{bmatrix} = \begin{bmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_{n-1} \\ \hat{c}_n \end{bmatrix} + \begin{bmatrix} \hat{v}_{1,1}(L) & \cdot & \cdot & \hat{v}_{1,n}(L) \\ \cdot & \cdot & \cdot & \cdot \\ \hat{v}_{n-1,1}(L) & \cdot & \cdot & \hat{v}_{n-1,n}(L) \\ \hat{v}_{n,1}(L) & \cdot & \cdot & \hat{v}_{n,n}(L) \end{bmatrix} \begin{bmatrix} x_{1,T+1} \\ \vdots \\ x_{n-1,T+1} \\ \pi_{T+1} \end{bmatrix}.$$

The expected real return on asset i in period $T+1$, the portfolio formation period, is

$$(7) \quad \begin{bmatrix} E_T r_{1,T+1} \\ \vdots \\ E_T r_{n-1,T+1} \end{bmatrix} = \begin{bmatrix} E_T x_{1,T+1} \\ \vdots \\ E_T x_{n-1,T+1} \end{bmatrix} + \begin{bmatrix} r_{TB,T+1}^n \\ \vdots \\ r_{TB,T+1}^n \end{bmatrix} - \begin{bmatrix} E_T \pi_{T+1} \\ \vdots \\ E_T \pi_{T+1} \end{bmatrix}$$

where $r_{TB,T+1}^n$ is the ex ante observed nominal return on Treasury bills for time $T+1$. The expected real return on Treasury bills for time $T+1$ is

$$(8) \quad E_T r_{TB,T+1} = r_{TB,T+1}^n - E_T \pi_{T+1}.$$

Finally, the conditional probability distribution for real returns for time $T+1$ is determined by

$$(9) \quad \begin{bmatrix} \tilde{r}_{1,T+1} \\ \cdot \\ \tilde{r}_{n-1,T+1} \\ \tilde{r}_{TB,T+1} \end{bmatrix} = \begin{bmatrix} E_T x_{1,T+1} \\ \cdot \\ E_T x_{n-1,T+1} \\ 0 \end{bmatrix} + \begin{bmatrix} r_{TB,T+1}^n \\ \cdot \\ r_{TB,T+1}^n \\ r_{TB,T+1}^n \end{bmatrix} - \begin{bmatrix} E_T \pi_{T+1} \\ \cdot \\ E_T \pi_{T+1} \\ E_T \pi_{T+1} \end{bmatrix} + \begin{bmatrix} \tilde{\varepsilon}_{1,T+1} \\ \cdot \\ \tilde{\varepsilon}_{n-1,T+1} \\ 0 \end{bmatrix} - \begin{bmatrix} \tilde{\varepsilon}_{\pi,T+1} \\ \cdot \\ \tilde{\varepsilon}_{\pi,T+1} \\ \tilde{\varepsilon}_{\pi,T+1} \end{bmatrix}$$

where $\begin{bmatrix} \tilde{\varepsilon}_{1,T+1} \\ \cdot \\ \tilde{\varepsilon}_{n-1,T+1} \\ \tilde{\varepsilon}_{\pi,T+1} \end{bmatrix}$ takes on the historically observed values $\begin{bmatrix} \tilde{\varepsilon}_{1,t} \\ \cdot \\ \tilde{\varepsilon}_{n-1,t} \\ \tilde{\varepsilon}_{\pi,t} \end{bmatrix}$ from regression (4),

$t=1,2,\dots,T$, with equal probabilities ($1/T$).

This method of deriving asset returns probability distribution functions, using historically occurring innovations to asset returns captured through this VAR procedure, is superior to the VAR methods mentioned in the literature, e.g. Campbell and Viceira (2002). The literature on derivation of asset returns probability distribution functions assumes that the distribution of asset returns is static, not evolving over time. But the reality is such that the asset returns distribution is dynamic, depending on both recent realizations and the fixed historical distribution of shocks to the dynamic asset returns process. So the right way of deriving asset returns probability distribution functions is to include the dynamics of the past history of asset returns.

D. Constrained Portfolios

Regulation T imposes on all investors, individual as well as institutional, 50% margin requirements. These requirements say that the ratio between the total position size (the sum of absolute values of portfolio holdings for all nominally risky assets) and initial wealth cannot be greater than two. This 50% margin restriction is implemented in the paper the following way.

Using the information about those randomly picked assets' derived probability distributions for their real returns (computed as shown in (9)), I compute the constrained optimal portfolio: the solution of

$$(10) \quad \underset{\{\alpha_1, \dots, \alpha_{n-1}\}}{\text{Max}} EU(\tilde{w}) = \text{Max} E \left\{ \frac{1}{\gamma} \left[w_0 (\alpha_1 \tilde{r}_1 + \dots + \alpha_{n-1} \tilde{r}_{n-1} + (1 - \alpha_1 - \dots - \alpha_{n-1}) \tilde{r}_{TB}) \right]^\gamma \right\}$$

subject to the short-selling constraint:

$$(11) \quad \sum_{i=1}^{n-1} |\alpha_i| \leq 2w_0$$

where w_0 is initial wealth, set equal to 1; $\alpha_1, \dots, \alpha_{n-1}$ are the first $n-1$ individual assets' portfolio shares in the constrained optimal portfolio. To get the portfolio I search over $\alpha_1, \dots, \alpha_{n-1}$ space to optimize expected utility, using nonlinear optimization by a quasi-Newton method based on convergence to first-order conditions of problem (10). Again, the expectation is taken over the joint probability distribution derived as described above in (4)-(9).

E. Unconstrained Portfolios

The next step, then, is to get the unconstrained optimal portfolio: the solution of

$$(12) \quad \underset{\{\beta_1, \dots, \beta_{n-1}\}}{\text{Max}} EU(\tilde{w}) = \text{Max} E \left\{ \frac{1}{\gamma} \left[w_0 (\beta_1 \tilde{r}_1 + \dots + \beta_{n-1} \tilde{r}_{n-1} + (1 - \beta_1 - \dots - \beta_{n-1}) \tilde{r}_{TB}) \right]^\gamma \right\}$$

where $\beta_1, \dots, \beta_{n-1}$ are the first $n-1$ individual assets portfolio shares in the unconstrained optimal portfolio. To get the portfolio I search over $\beta_1, \dots, \beta_{n-1}$ space to optimize expected utility, using nonlinear optimization by a quasi-Newton method based on convergence to first-order conditions of problem (12). Again, the expectation is taken over the joint probability distribution derived as described above in (4)-(9).

F. Calculating Opportunity Cost

With the constrained optimal and unconstrained optimal portfolios constructed I calculate the opportunity cost, θ . For the formula for θ , (3), $E(\tilde{R}^\gamma)^{\text{unconstrained}}$ and $E(\tilde{R}^\gamma)^{\text{constrained}}$ are calculated as follows:

$$(13) \quad E_T(\tilde{R}_{T+1}^\gamma)^{\text{unconstrained}} \text{ (referring more completely to } E_T(\tilde{R}_{T+1}^\gamma)^{\text{unconstrained}} \text{) is}$$

$$= \frac{1}{T} \sum_{t=1}^T \left\{ \begin{bmatrix} \beta_1^* & \dots & \beta_{n-1}^* & 1 - \beta_1^* - \dots - \beta_{n-1}^* \end{bmatrix} \begin{bmatrix} E_T r_{1,T+1} + \varepsilon_{1,t} - \varepsilon_{\pi,t} \\ \cdot \\ E_T r_{n-1,T+1} + \varepsilon_{n-1,t} - \varepsilon_{\pi,t} \\ E_T r_{TB,T+1} + 0 - \varepsilon_{\pi,t} \end{bmatrix} \right\}^\gamma$$

where the vector of β_i^* is the vector of optimal unconstrained portfolio shares; the vector of $E_T r_{i,T+1} + \varepsilon_{i,t} - \varepsilon_{\pi,t}$ and $E_T r_{TB,T+1} - \varepsilon_{\pi,t}$ are the vectors of particular possible values of real returns (conditional on the data set for times $t=1$ through T) at time $T+1$ (the portfolio formation period) and calculated as shown in (4)-(9).

And $E(\tilde{R}^\gamma)^{\text{constrained}}$ is

$$(14) \quad E_T(\tilde{R}_{T+1}^\gamma)^{\text{constrained}} = \frac{1}{T} \sum_{t=1}^T \left\{ \begin{bmatrix} \alpha_1^* & \dots & \alpha_{n-1}^* & 1 - \alpha_1^* - \dots - \alpha_{n-1}^* \end{bmatrix} \begin{bmatrix} E_T r_{1,T+1} + \varepsilon_{1,t} - \varepsilon_{\pi,t} \\ \cdot \\ E_T r_{n-1,T+1} + \varepsilon_{n-1,t} - \varepsilon_{\pi,t} \\ E_T r_{TB,T+1} + 0 - \varepsilon_{\pi,t} \end{bmatrix} \right\}^\gamma$$

where α_i^* is the vector of optimal constrained portfolio shares.

Then, having calculated (13) and (14), I use (3) to get a numerical value for θ .

The whole procedure, starting from picking $n-1$ nominally risky assets, is being repeated 1,000 times. This gives 1,000 values of θ . The procedure is done for each of 11

alternative values of the risk aversion parameter γ each time for the well-diversified number of assets, n , and for ten assets.

IV. Results

The results from this research project are as follows.

A. Results Derived from Historical Returns Data with No Exaggeration of Extreme Returns.

A.1. Opportunity Costs

Table 1 and Table 2 represent the results from calculation of 1,000 values of the proportionate opportunity cost for 11 different values of relative risk aversion for alternatively the well-diversified number of assets for each level of risk aversion and for ten assets.

Both tables clearly show that as the level of relative risk aversion increases the proportionate opportunity cost decreases, given the CRRA utility function, (1). This is not surprising. As risk aversion decreases the investor considers the optimal unconstrained portfolio as his best choice that does not place any restrictions on his investment behavior and lets him follow a very aggressive short sale strategy that will not be possible under the constrained portfolio strategy (see Table 3 and Table 4 below), and, therefore, he will require a higher proportion of initial wealth as the payment to stay constrained and be indifferent to the optimal constrained portfolio with the short-selling restriction.

Table 1

The proportionate opportunity cost of the short-selling constraint, $(\theta - 1)$, for the well-diversified number of assets

Table 2

The proportionate opportunity cost of the short-selling constraint, $(\theta - 1)$, for ten assets

Table 1 and Table 2 also show that as the level of relative risk aversion increases the standard deviations of the proportionate opportunity costs decrease: the distributions of the opportunity cost are getting “tighter”. This can be explained by the following: as risk aversion increases, first, investors become less and less engaged into short-selling due to a decrease in their risk-tolerance; therefore, at some point the short-selling constraint stops binding. Second, a bigger and bigger proportion of initial wealth will be placed by the investors into Treasury bills (see Table 3 and Table 4), the least shorted asset from their point of view, and a smaller and smaller proportion of initial wealth will be placed into nominally risky assets, the most shorted assets. The combination of these two facts results in portfolios that very much resemble each other.

Thus we observe the low standard deviation of the proportionate opportunity cost. For the well-diversified number of assets, Table 1 shows that the lowest mean (over 1,000 replications) of the proportionate opportunity cost, 0.0% (0.000), corresponds to the level of risk aversion of 29 and higher. This means that an investor with the level of relative risk aversion of 29 and higher being unconstrained will be equally happy as if he

was constrained. The highest mean (over 1,000 replications) of the proportionate opportunity cost, 12.8% (0.128), corresponds to the very low level of relative risk aversion of 0.7. This means that an investor with the level of relative risk aversion of 0.7 being unconstrained will be equally happy as if he was constrained but had 12.8% more of initial wealth.

For low levels (from 3 to 0.7) of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost ranges from 4.0% (0.040) for the relative risk aversion of 3 to 12.8% (0.128) for the relative risk aversion of 0.7.

For medium (from 12 to 9) levels of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost ranges from 0.3% (0.003) for the relative risk aversion of 12 to 1.0% (0.010) for relative risk aversion of 9. This suggests that medium risk-tolerance investors also value optimal unconstrained portfolios but only to some extent, and require from 0.3% to 1.0% of additional initial wealth to stay constrained.

For high (from 29 to 31) levels of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost is 0.0% (0.000) in each case. This suggests that low risk-tolerance investors value optimal unconstrained portfolios as well as optimal constrained portfolios and do not require any additional payment to stay indifferent to being constrained.

For ten assets, Table 2 shows that the lowest mean (over 1,000 replications) of the proportionate opportunity cost, 0.0% (0.000), corresponds to the high level of risk aversion of 31. This means that an investor with the level of relative risk aversion of 31 being unconstrained will be equally happy as if he was constrained. The highest mean

(over 1,000 replications) of the proportionate opportunity cost, 8.0% (0.080), corresponds to the very low level of relative risk aversion of 0.7. This means that an investor with the level of relative risk aversion of 0.7 being unconstrained will be equally happy as if he was constrained but had 8.0% more of initial wealth.

The highest values of the proportionate opportunity cost (the means over 1,000 replications) correspond to low levels of relative risk aversion (from 0.7 to 3) and range from 2.1% (0.021) for the relative risk aversion of 3 to 8.0% (0.080) for the relative risk aversion of 0.7. Investors with low levels of risk aversion (from 3 to 0.7) in the presence of ten assets will require from 2.1% to 8.0% of initial wealth to stay constrained and being indifferent to the restrictions on short-selling. These magnitudes, as a matter of fact, are lower than those for the well-diversified number of assets in Table 1.

For medium (from 12 to 9) levels of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost is 0.6% (0.006) for risk aversion of 9 and 10, and 0.5% (0.005) for risk aversion of 11 and 12. These numbers are close to those for the well-diversified number of assets in Table 1. The explanation is that the well-diversified number of assets for medium levels of risk aversion is close to ten (see Table 3 and Table 4); therefore the proportionate opportunity cost in both cases will be about the same.

For high (from 29 to 31) levels of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost is 0.1% (0.001) for risk aversion of 29 and 31; and 0.0% (0.000) for the risk aversion of 31.

A.2. Optimal Portfolio Shares

Table 3 and Table 4 present typical optimal portfolio shares for unconstrained and constrained portfolio strategies for three different levels of relative risk aversion: low (of 0.7), medium (of 11) and high (of 31), for the well-diversified number of assets and for

Table 3

Illustrative optimal portfolio shares for unconstrained portfolios and portfolios with the short-selling constraint for the well-diversified number of assets

Table 4

Illustrative optimal portfolio shares for unconstrained portfolios and portfolios with the short-selling constraint for ten assets¹

ten assets, in each case for a different set of available assets giving an opportunity cost that is typical for that level of risk aversion.

For both tables for unconstrained portfolios for risk aversion of 0.7, more than 100% of initial wealth, w_0 , is held in the nominally risky assets (asset #1 through asset #23 in Table 3 and asset #1 through asset #9 in Table 4) as a group, and Treasury bills are held in negative quantities.

As risk aversion increases, the proportion of initial wealth held in Treasury bills first, reaches zero, and then becomes positive; and correspondingly the proportion of initial wealth held in the group of nominally risky assets decreases.

The tables show that unconstrained optimal portfolio shares are not similar for different levels of risk aversion. As a matter of fact, optimal unconstrained portfolios for the low level of relative risk aversion of 0.7 have more extreme quantities (negative as

well as positive) of assets than optimal unconstrained portfolios for medium level of relative risk aversion of 11 and for high level of relative risk aversion of 31. Extremely negative quantities of assets for high risk-tolerance investors mean that the investors follow a very aggressive short-sale strategy.

The tables also show that constrained optimal portfolio shares are not similar for different levels of risk aversion as well. Optimal constrained portfolios for the low level of relative risk aversion of 0.7 have more extreme quantities (negative as well as positive) of assets than optimal constrained portfolios for medium level of relative risk aversion of 11 and for high level of relative risk aversion of 31.

Also Table 3 and Table 4 show monthly expected returns on unconstrained and constrained optimal portfolios, $E(X^* \tilde{R})$, for the three levels of relative risk aversion (0.7, 11 and 31).

The expected returns for constrained and unconstrained optimal portfolios for risk aversion of 0.7 are very dramatic for the well-diversified portfolios and large for ten-asset portfolios. Expected returns are of large size for risk aversion of 11 and of small size for risk aversion of 31. Such extreme magnitudes of expected portfolio returns for high risk-tolerance investors confirm the previously made conclusion about very aggressive short-sale strategies. These magnitudes represent very leveraged portfolios. For investors with risk aversion of 11 and 31 there is, definitely, some short-selling is going on too, but not as aggressive as for investors with risk aversion of 0.7. The less aggressive short-selling for medium or high risk aversion leads to lower mean portfolio returns.

The big difference between the expected portfolio returns with the well-diversified number of assets and those with ten assets for the risk aversion of 0.7 is due to

the level of diversification, which is optimal in the first case and sub-optimal in the second.

In comparing unconstrained expected portfolio returns and constrained expected portfolio returns for the three levels of risk aversion for the two tables I find that unconstrained and constrained expected portfolio returns for risk aversion of 31 are the same; for risk aversion of 11 they are close; for risk aversion of 0.7 unconstrained and constrained expected portfolio returns are not close at all. These unconstrained and constrained expected portfolio returns show that as risk aversion increases, the closer to each other expected returns on unconstrained and constrained portfolios are, and thus the more nearly indifferent an investor is between the unconstrained and constrained portfolio strategies. This happens due to the fact that as risk aversion increases, investors become less and less involved in short-selling; therefore, the short-selling constraint finally is not binding at all, and optimal unconstrained and constrained portfolio strategies more and more resemble each other.

Table 3 and Table 4 also report the *certainty equivalents* calculated for the same three levels of relative risk aversion (0.7, 11 and 31). The *certainty equivalent*, (CE), is defined by

$$(15) \quad \frac{1}{\gamma} CE^\gamma = \frac{1}{\gamma} w_0^\gamma E(\tilde{R}^\gamma)$$

and so, with $w_0=1$,

$$(16) \quad CE = \left(E[\tilde{R}^\gamma] \right)^{\frac{1}{\gamma}}.$$

The certainty equivalent represents the amount of certain wealth that would be viewed with indifference to the optimal portfolio. It is computed for investors of different

levels of risk aversion: low (of 0.7), medium (of 11) and high (of 31). The two tables show that as risk aversion increases the value of the certainty equivalent decreases (for the unconstrained portfolio strategy as well as for the constrained). This suggests that as investors become more afraid of risk they use less risky portfolio strategies and will be expecting lower returns from those portfolios and, so, the certain amount of wealth they will be willing to accept with indifference will decrease.

It is interesting to compare certainty equivalents for unconstrained optimal portfolio strategies from Table 3 and Table 4 and the mean values of the proportionate opportunity cost from Table 1 and Table 2. For the well-diversified number of assets, as Table 1 shows, the mean over 1,000 replications of the proportionate opportunity cost reaches 12.8% for unconstrained investors with the level of risk aversion of 0.7 whose net certainty equivalent (the certainty equivalent minus 1.0) equals 18.7%. As the level of risk aversion increases to 31 the mean of the proportionate opportunity cost falls to 0.0% while the net certainty equivalent falls to -0.1%.

For ten assets, as Table 2 shows, the mean value of the proportionate opportunity cost reaches 8.0% for unconstrained investors with their net certainty equivalent being equal to 13.3% for the risk aversion of 0.7. As the level of risk aversion increases to 31 the mean of the proportionate opportunity cost for unconstrained investors falls to 0.0% while the net certainty equivalent falls to 0.0%.

A.3. Regret in the Worst-Case Scenario

Large negative and positive asset holdings (Table 3 and Table 4) in unconstrained portfolios and to some extent in constrained portfolios for investors with a level of risk

aversion of 0.7 suggest that the investors take on a lot of risk. This raises the question: if the worst possible portfolio outcome occurs, then how much will the investors suffer from such an outcome? It is possible to measure the investors' proportionate regret from the worst-case scenario with such a risky portfolio.

Table 5 and Table 6 report the proportionate regret, $(\theta - 1)$, for the well-diversified number of assets and for ten assets, that will be incurred by investors if the worst possible outcome of asset returns occurs. This θ is defined by

$$(17) \quad U(\theta (X^{*'} R)^{worst}) = EU(X^{*'} \tilde{R})$$

where X^* is the optimally chosen portfolio, $(X^{*'} R)^{worst}$ is the one of the 120 states of nature giving the lowest portfolio return, $U[(X^{*'} R)^{worst}]$ is an investor's utility from getting the worst possible portfolio outcome, $EU(X^{*'} \tilde{R})$ is an investor's ex ante expected utility.

For unconstrained investors (for the case with the well-diversified number of assets as well as for ten assets) the mean of the proportionate regret (over 1,000 replications) is the highest for the low level of risk aversion of 0.7 and the lowest for the high level of risk aversion of 31. This means that high risk-tolerance investors do choose very risky unconstrained asset allocations. And it is riskier when the number of assets is

Table 5

The ex post proportionate regret, $(\theta - 1)$, for the well-diversified number of assets

Table 6

The ex post proportionate regret, $(\theta - 1)$, for ten assets

at the well-diversified level. Those asset allocations are so risky at the level of risk aversion of 0.7, that if the worst possible outcome occurs it would require for investors with the well-diversified number of assets to receive 1021.0% of initial wealth in compensation and for investors with ten assets to receive 641.2% of initial wealth in order to get the same level of ex post utility as their ex ante expected utility. For the high level of 31 for risk aversion the mean of the proportionate regret (over 1,000 replications) is 2.0% (0.020) for investors with well-diversified number of assets and 2.8% (0.028) for investors with ten assets. Such a low proportionate regret suggests that low risk-tolerance unconstrained investors choose very conservative unconstrained asset allocations. So conservative are their allocations that even the worst possible outcome will require for them less than 3.0% of initial wealth to get to the same level of utility as their ex ante expected utility.

For constrained portfolio strategies the mean proportionate regret (over 1,000 replications) ranges from 110.9% (1.109) for risk aversion of 0.7 to 2.0% (0.020) for risk aversion of 31 for investors with the well-diversified number of assets, and from 100.9% (1.009) for risk aversion of 0.7 to 2.8% (0.028) for risk aversion of 31 for investors with ten assets. This means that constrained portfolios have a very restrictive character and do not let high risk-tolerance investors take nearly as much risk as they would in the absence of the constraint. For low risk-tolerance investors constrained portfolios are virtually the same as unconstrained portfolios (the difference in the mean regret is 0.0%) and represent very conservative asset allocations with very little risk to take. Note that the tendency for the short-selling constraint to make portfolios more conservative is also seen in Table 3

and Table 4, which show that at each level of risk aversion the mean portfolio return is less when the constraint is present than when it is not.

B. Historical Data with Extreme Values Exaggerated

To check the robustness of the estimates of the proportionate opportunity cost I include extremely high and extremely low simulated asset returns in each data set of available assets (well-diversified portfolios as well as ten-asset portfolios).

The simulated extremely high and extremely low asset returns are constructed the following way. For the original data set for each historical time period I compute the average excess return across all assets in the data set. A historical time period with the highest average excess return across all assets defines the historical period with the highest returns. A historical time period with the lowest average excess return across all assets defines the historical period with the lowest returns. Then, for the extreme historical periods only the deviation of each asset's return from that asset's intertemporal mean return is calculated. The deviations are doubled and then added back to assets' intertemporal means. This way I create two fictional time periods with exaggerated high and exaggerated low returns. These fictional asset returns provide simulated extreme time periods to replace the time periods they were constructed from. The rest of the original data set remains unchanged.

Then I repeat the whole procedure of calculating the proportionate opportunity cost 1,000 times in each case simulating two fictional data periods. This gives 1,000 new θ 's from the 1,000 data sets with extreme returns exaggerated. The results from this

project are reported in Table 7 for the well-diversified number of assets and Table 8 for ten assets.

Out of all values considered for relative risk aversion, the lowest mean (over 1,000 replications) of the proportionate opportunity cost for the well-diversified number of assets and for ten assets corresponds to the high level of relative risk aversion of 31. The highest mean (over 1,000 replications) of the proportionate opportunity cost for the well-diversified number of assets and for ten assets corresponds to the low level of relative risk aversion of 0.7.

The results in the second project confirm the results from the first one: as the level of relative risk aversion increases the proportionate opportunity cost decreases, the better the constrained portfolio performs ex ante and the lower the proportion of initial wealth an investor requires to stay constrained and accept the short-selling restriction. Also, the same way as with the original data set, as the level of relative risk aversion increases the standard deviation of the proportionate opportunity cost decreases. This means that as the level of risk aversion increases, as investors facing different asset sets become less risk tolerant, their perceptions of the optimal constrained portfolio strategy, even with extreme historical periods, are more similar to each other than perceptions of the strategy for investors with lower risk aversion.

The only difference between Tables 1-2 and Tables 7-8 is that the magnitude of the proportionate opportunity cost is bigger when extreme returns exaggerated. These differences can be explained by the following. The calculation of the proportionate opportunity cost in Tables 7 and 8 was based on the historical asset returns distribution with the extreme returns exaggerated. This exaggeration of historically occurred extreme

Table 7

The proportionate opportunity cost of the short-selling constraint, $(\theta - 1)$, with extreme returns exaggerated for the well-diversified number of assets

Table 8

The proportionate opportunity cost of the short-selling constraint, $(\theta - 1)$, with extreme returns exaggerated for ten assets

returns converted the original asset returns distribution, derived as shown in (4)-(9), into a distribution with fatter tails. This change in the distribution increases the probability for investors to end up with extremely low portfolio returns. Hence, the proportionate opportunity cost of using the constrained optimal portfolio instead of the unconstrained one in the case of exaggeration of extreme returns will be higher.

The lowest mean (over 1,000 replications) of the proportionate opportunity cost for the well-diversified number of assets (Table 7) is 0.0% (0.000) and corresponds to the level of risk aversion of 31. This means that an investor with the level of relative risk aversion of 31 being unconstrained will be equally happy as if he was constrained. The highest mean (over 1,000 replications) of the proportionate opportunity cost, 13.5% (0.135), corresponds to the very low level of relative risk aversion of 0.7. This means that an investor with the level of relative risk aversion of 0.7 being unconstrained will be equally happy as if he was constrained but had 13.5% more of initial wealth.

The lowest mean (over 1,000 replications) of the proportionate opportunity cost for ten assets (Table 8) is 0.1% (0.001) and corresponds to the level of risk aversion of 29 and higher. The highest mean (over 1,000 replications) of the proportionate opportunity cost, 9.1% (0.091), corresponds to the very low level of relative risk aversion of 0.7.

V. Conclusion

In this paper I have investigated the opportunity cost incurred by investors when they use optimal portfolios constrained by a short-selling restriction instead of unconstrained optimal portfolios. Two sets of returns have been used: the original historical asset returns and historical asset returns with extreme values exaggerated. CRRA utility functions and the proportionate opportunity cost have been used. The opportunity cost has been calculated for different values of relative risk aversion (including extreme levels of relative risk aversion) for the well-diversified portfolios and ten-asset portfolios. The highest mean across simulations of the proportionate opportunity cost found is 13.5% (0.135) for the level of relative risk aversion of 0.7 with extreme returns exaggerated for the well-diversified optimal portfolios. The lowest mean of the proportionate opportunity cost found is 0.0% (0.000) for the level of relative risk aversion of 29 and higher for the original historical asset returns for the well-diversified optimal portfolios and for risk aversion of 31 for the ten-asset portfolio. For both data sets, for the original historical asset returns and for historical asset returns with extreme values exaggerated, as the level of relative risk aversion increases the proportionate opportunity cost decreases.

The only difference between estimates of the proportionate opportunity cost for these two data sets is the magnitude of the estimates. They are bigger for the data sets when extreme returns exaggerated. This can be explained by the fact that the presence of extreme returns in the probability distribution creates fatter tails in the distribution which increases the opportunity cost of accepting the constraint.

Therefore, based on my calculations, I may conclude that for investors with low levels of risk aversion the short-selling constraint represents a serious hazard. For investors with medium levels of relative risk aversion the constrained portfolio strategy performs quite well. For investors with very high levels of relative risk aversion (29 and above) the optimal portfolio strategy with the short-selling constraint performs as well as the unconstrained portfolio strategy.

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Table 1

The proportionate opportunity cost of the short-selling constraint, $(\theta - I)$, for various values of relative risk aversion for the well-diversified number of assets

<i>Relative Risk Aversion, $(1-\gamma)$</i>	<i># of Assets</i>	<i>Smallest</i>	<i>Mean</i>	<i>Median</i>	<i>Largest</i>	<i>Standard Deviation</i>
Low						
0.7	24	0.043	0.128	0.121	0.583	0.046
1	23	0.022	0.086	0.074	0.556	0.032
2	22	0.007	0.068	0.055	0.486	0.013
3	20	0.001	0.040	0.029	0.444	0.006
Medium						
9	9	0.000	0.010	0.000	0.193	0.004
10	8	0.000	0.008	0.000	0.172	0.004
11	7	0.000	0.004	0.000	0.154	0.003
12	6	0.000	0.003	0.000	0.148	0.003
High						
29	3	0.000	0.000	0.000	0.003	0.001
30	3	0.000	0.000	0.000	0.001	0.001
31	3	0.000	0.000	0.000	0.001	0.001

Table 2

The proportionate opportunity cost of the short-selling constraint, $(\theta - I)$, for various values of relative risk aversion for ten assets

<i>Relative Risk Aversion, $(1-\gamma)$</i>	<i>Smallest</i>	<i>Mean</i>	<i>Median</i>	<i>Largest</i>	<i>Standard Deviation</i>
Low					
0.7	0.003	0.080	0.061	0.498	0.045
1	0.001	0.056	0.047	0.461	0.025
2	0.001	0.033	0.030	0.413	0.019
3	0.000	0.021	0.011	0.384	0.009
Medium					
9	0.000	0.006	0.000	0.185	0.006
10	0.000	0.006	0.000	0.156	0.004
11	0.000	0.005	0.000	0.126	0.003
12	0.000	0.005	0.000	0.128	0.003
High					
29	0.000	0.001	0.000	0.004	0.001
30	0.000	0.001	0.000	0.002	0.001
31	0.000	0.000	0.000	0.002	0.001

Table 3

Illustrative optimal portfolio shares for unconstrained portfolios and portfolios with the short-selling constraint for different values of relative risk aversion for the well-diversified number of assets¹

# of An Asset	Relative Risk Aversion, $(1-\gamma)$, equal to 0.7		Relative Risk Aversion, $(1-\gamma)$, equal to 11		Relative Risk Aversion, $(1-\gamma)$, equal To 31	
	Unconstrained	Constrained	Unconstrained	Constrained	Unconstrained	Constrained
1	0.456	0.006	0.025	0.019	0.151	0.153
2	-0.103	0.003	-0.103	0.025	0.007	0.007
3	-0.104	-0.001	0.824	-0.000	-	-
4	1.634	0.082	-1.777	-0.446	-	-
5	0.749	0.002	1.812	1.484	-	-
6	0.219	0.004	0.589	0.026	-	-
7	0.559	-0.003	-	-	-	-
8	-0.678	0.001	-	-	-	-
9	7.869	0.002	-	-	-	-
10	-0.985	-0.002	-	-	-	-
11	-1.786	+0.000	-	-	-	-
12	-1.614	0.004	-	-	-	-
13	-1.371	0.003	-	-	-	-
14	-1.574	-0.002	-	-	-	-
15	3.524	0.001	-	-	-	-
16	-0.389	-0.001	-	-	-	-
17	0.579	0.113	-	-	-	-
18	-0.861	0.189	-	-	-	-
19	0.749	0.004	-	-	-	-
20	-0.466	-0.004	-	-	-	-
21	3.269	0.458	-	-	-	-
22	2.426	0.793	-	-	-	-
23	-0.497	0.001	-	-	-	-
24 ²	-10.606	-0.653	-0.371	-0.108	0.842	0.840
$E(X^* \tilde{R})^3$	1.306	1.116	1.069	1.051	1.000	1.000
Certainty Equivalent	1.187	1.052	1.044	1.040	0.999	0.999

¹ Numbers are not comparable across levels of risk aversion, because for each level of risk aversion a different set of available assets was used: a set giving an exact value of opportunity cost typical for that level of risk aversion.

² The 24th asset is risk-free in nominal terms.

³ Monthly gross expected returns on portfolios.

Table 4

Illustrative optimal portfolio shares for unconstrained portfolios and portfolios with the short-selling constraint for different values of relative risk aversion for ten assets¹

# of An Asset	Relative Risk Aversion, $(1-\gamma)$, equal to 0.7		Relative Risk Aversion, $(1-\gamma)$, equal to 11		Relative Risk Aversion, $(1-\gamma)$, equal to 31	
	Unconstrained	Constrained	Unconstrained	Constrained	Unconstrained	Constrained
1	0.707	0.626	0.132	0.109	0.010	0.010
2	6.403	+0.000	0.814	0.522	-0.007	-0.007
3	-2.541	+0.000	0.263	0.225	-0.004	-0.004
4	-0.249	-0.000	-0.485	-0.222	-0.028	-0.028
5	2.174	+0.000	0.130	0.106	0.011	0.011
6	-2.407	-0.001	0.539	0.403	0.031	0.031
7	2.050	+0.000	0.231	0.117	0.008	0.008
8	-0.657	-0.004	0.196	0.168	0.014	0.014
9	3.648	1.368	-0.267	-0.128	0.118	0.118
10 ²	-8.129	-0.999	-0.553	-0.301	0.846	0.846
$E(X^* \tilde{R})^3$	1.243	1.102	1.047	1.034	1.002	1.002
Certainty Equivalent	1.133	1.049	1.020	1.015	1.000	1.000

¹ Numbers are not comparable across levels of risk aversion because for each level of risk aversion a different set of available assets was used: a set giving an exact value of opportunity cost typical for that level of risk aversion.

² The 10th asset is risk-free in nominal terms.

³ Monthly gross expected returns on portfolios.

Table 5

The ex post proportionate regret, $(\theta - I)$, under the worst portfolio outcome for the well-diversified number of assets

Relative Risk Aversion, $(1-\gamma)$	Portfolios	# of Assets	Smallest	Mean	Median	Largest	Standard Deviation
0.7	<i>Unconstrained</i>	24	2.628	10.210	9.244	31.444	4.168
	<i>Constrained</i>	24	0.309	1.109	0.902	4.331	0.486
11	<i>Unconstrained</i>	7	0.029	0.077	0.069	0.614	0.047
	<i>Constrained</i>	7	0.029	0.076	0.069	0.386	0.037
31	<i>Unconstrained</i>	3	0.004	0.020	0.015	0.163	0.017
	<i>Constrained</i>	3	0.004	0.020	0.015	0.127	0.016

Table 6

The ex post proportionate regret, $(\theta - I)$, under the worst portfolio outcome for ten assets

Relative Risk Aversion, $(1-\gamma)$	Portfolios	Smallest	Mean	Median	Largest	Standard Deviation
0.7	<i>Unconstrained</i>	0.345	6.412	5.015	18.159	2.696
	<i>Constrained</i>	0.227	1.009	0.849	2.742	1.269
11	<i>Unconstrained</i>	0.034	0.083	0.079	0.608	0.035
	<i>Constrained</i>	0.034	0.082	0.079	0.327	0.026
31	<i>Unconstrained</i>	0.014	0.028	0.027	0.098	0.009
	<i>Constrained</i>	0.014	0.028	0.027	0.094	0.009

Table 7

The proportionate opportunity cost of the short-selling constraint, $(\theta - I)$, for various values of relative risk aversion with extreme returns exaggerated for the well-diversified number of assets

Relative Risk Aversion, $(1-\gamma)$	# of Assets	Smallest	Mean	Median	Largest	Standard Deviation
Low						
0.7	25	0.047	0.135	0.129	0.586	0.061
1	24	0.031	0.091	0.089	0.570	0.049
2	23	0.010	0.073	0.068	0.487	0.025
3	21	0.005	0.061	0.053	0.439	0.012
Medium						
9	11	0.001	0.016	0.000	0.199	0.005
10	10	0.001	0.010	0.000	0.181	0.005
11	9	0.000	0.009	0.000	0.163	0.004
12	8	0.000	0.006	0.000	0.150	0.003
High						
29	4	0.000	0.001	0.000	0.004	0.001
30	3	0.000	0.001	0.000	0.003	0.001
31	3	0.000	0.000	0.000	0.003	0.001

Table 8

The proportionate opportunity cost of the short-selling constraint, $(\theta - I)$, for various values of relative risk aversion with extreme returns exaggerated for ten assets

Relative Risk Aversion, $(1-\gamma)$	Smallest	Mean	Median	Largest	Standard Deviation
Low					
0.7	0.029	0.091	0.121	0.516	0.051
1	0.016	0.069	0.080	0.509	0.039
2	0.006	0.047	0.061	0.453	0.022
3	0.003	0.033	0.031	0.419	0.010
Medium					
9	0.001	0.014	0.000	0.160	0.006
10	0.001	0.009	0.000	0.154	0.005
11	0.000	0.007	0.000	0.143	0.005
12	0.000	0.005	0.000	0.131	0.004
High					
29	0.000	0.001	0.000	0.004	0.001
30	0.000	0.001	0.000	0.003	0.001
31	0.000	0.001	0.000	0.003	0.001