THE OPPORTUNITY COST FOR AN INVESTOR OF BEING CONSTRAINED BY THE TYPE OF ASSETS IN HIS PORTFOLIO: BONDS ONLY OR STOCKS ONLY

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Abstract

Restricting the type of assets in an investor's portfolio results in a welfare loss for the investor. In this paper I explore investors' welfare losses when they restrict themselves to invest in either stocks or bonds but not both. The restriction gives investors sub-optimal asset allocations that result in welfare losses for the investors. To measure those welfare losses I compare "only stock indices and Treasury bills" optimal portfolios or "only bond indices and Treasury bills" optimal portfolios with "stock and bond indices and Treasury bills" optimal portfolios using the concept of the proportionate opportunity cost along with various CRRA utility functions. The original historical asset returns data set is used with a VAR in generating joint returns distributions for the portfolio formation period. I show that for investors with low levels of risk aversion welfare losses do not exceed 1.5% of initial wealth when they invest sub-optimally. For investors with medium and high levels of relative risk aversion, constrained portfolios that include only one type of assets, stocks only or bonds only, along with Treasury bills, give expected utility about as high as unconstrained portfolios that include both types of assets, stocks and bonds.

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1. Introduction

The key question in the literature regarding well-diversified portfolios is: will one-type-asset portfolios be well-diversified? In other words, will one diversify sub-optimally if one invests in a portfolio that consists of stocks only, or bonds only as opposed to a portfolio that consists of both stocks and bonds?

One motivation for doing this case comes from Haliassos and Bertaut (1995). The authors investigated why many people do not hold stocks in their portfolios. In the article they pointed out that: "…[b]etween 75% and 80% of United States households … do not hold stocks directly. This proportion is remarkably stable through time and across data bases". They suggested several reasons to explain the puzzle (e.g. costly information concerning the process of investing in stocks, education, cultural factors) and found support for their ideas through empirical studies.

But the question that remains unanswered in the article and the question I am interested is: how big is one's welfare loss if one restricts oneself to only one type of assets, i.e. bonds only or stocks only, rather than to have a portfolio of bonds and stocks together.

In order to answer the question I will compare expected utility from the optimal portfolio constrained to include bonds only or stocks only with that from the optimal unconstrained portfolio permitted to have both bonds and stocks, by using the concept of opportunity cost (Brennan and Torous (1999), Tew, Reid and Witt (1991)).

The proportionate opportunity cost is the best way to measure investors' welfare losses because the results are readily interpretable as intuitively "large" or "small", which would not be true if compensating payments were expressed in additive dollar terms. Under the assumption of the constant relative risk aversion utility function

(1) $$\mathrm{U}(\tilde{w}) = \begin{cases} \dfrac{1}{\gamma}\tilde{w}^{\gamma}, \ \gamma < 1, \ \gamma \neq 0, \ \tilde{w} > 0 \\ -\infty, \ \tilde{w} \leq 0 \end{cases}$$

the proportionate opportunity cost (willingness to accept payment as compensation for being constrained to only one type of assets) can be calculated as θ - *1.0* where θ is defined by

(2) $$\mathrm{EU}(\theta w_0 \, \tilde{R}^{optimal}_{Bonds}) = \mathrm{EU}(w_0 \, \tilde{R}^{optimal}_{Bonds\ and\ Stocks})$$

and

(3) $$\mathrm{EU}(\theta w_0 \, \tilde{R}^{optimal}_{Stocks}) = \mathrm{EU}(w_0 \, \tilde{R}^{optimal}_{Bonds\ and\ Stocks})$$

where w_0 is the initial wealth, $\tilde{R}^{optimal}_{Bonds\ and\ Stocks}$, $\tilde{R}^{optimal}_{Bonds}$, and $\tilde{R}^{optimal}_{Stocks}$ are the stochastic returns per dollar invested for the portfolios with both bonds and stocks, for the portfolio with bonds only, and for the portfolio with stocks only. Solving (2) and (3) with the utility function (1) gives

(4) $$\theta = \left[\frac{E(\tilde{R}^{\gamma}_{Bonds\ and\ Stocks})^{optimal}}{E(\tilde{R}^{\gamma}_{Bonds})^{optimal}} \right]^{\frac{1}{\gamma}}$$

and

(5) $$\theta = \left[\frac{E(\tilde{R}^{\gamma}_{Bonds\ and\ Stocks})^{optimal}}{E(\tilde{R}^{\gamma}_{Stocks})^{optimal}} \right]^{\frac{1}{\gamma}}.$$

Under CRRA θ also equals the ratio of certainty equivalents of the bonds-and-stocks unconstrained and bonds/stocks only constrained optimal portfolios. Since the ratio of certainty equivalents is unitless, and in particular has no time units, the proportionate opportunity cost, θ-*1.0*, is also timeless. But its numerical value depends on a number of months until horizon, i.e. with the investment horizon of T months the proportionate willingness to accept payment to accept the constraint is θ^T.

The procedure used in this paper, for calculating the proportionate opportunity cost for an investor of being constrained by the type of assets in his portfolio, includes estimation of a vector autoregressive process, derivation of the joint probability distribution function of asset returns, and computing optimal portfolios.

To generate the ex ante returns distribution for the investment period I use a vector autoregressive process (VAR) to project the means of returns and to capture 120 historically occurring shocks to all asset returns; then I assume that the true distribution of shocks for the investment period is given by adding those 120 sets of returns shocks with equal probabilities to the conditional means.

In this paper I show that for investors with high levels of relative risk aversion (nine and above), constrained portfolios that include only one type of assets, bonds only or stocks only, along with Treasury bills perform as well as unconstrained portfolios that include both types of assets, bonds and stocks.

The second section of the paper describes the procedure of forming investors' portfolios, of inferring the joint probability distribution function of asset returns via a vector autoregression, of computing the constrained optimal and unconstrained optimal portfolios, and of the calculation of the proportionate opportunity cost.

The third section discusses the results. The fourth section of the paper concludes and summarizes.

2. The Procedure

2.1. Portfolio Formation

To form the "only stocks" constrained portfolio I use two composite stock indices: S&P 500 and NASDAQ, with Treasury bills as the nominally risk-free asset. To form the "only bonds" constrained portfolio I use two composite bond indices: Salomon Brothers' Long-Term High-Grade Corporate Bonds Index and Long-Term Government Bonds Total Return index, with Treasury bills as the nominally risk-free asset. The unconstrained portfolio includes both types of assets: stocks and bonds. To form the unconstrained portfolio I use the same two composite stock indices and the same two composite bond indices, and Treasury bills.

2.2. Vector Autoregressions of Returns

To get expected values and probability distributions of real returns for the four indices and Treasury bills at time $T+1$, the portfolio formation period, I estimate a vector autoregressive process (VAR). Then I derive the joint probability distribution for the four indices and Treasury bills real returns, and, finally, I construct optimal constrained and optimal unconstrained portfolios.

To derive the joint probability distribution of empirical deviations from the VAR-estimated conditional means for those four indices returns and inflation I do the following.

The nominal return on index i at time t minus the nominal return on Treasury bills at time t gives us the excess return on index i ($x_{i,t}$) at time t for $i=1,\ldots,4$ and for $t=1, \ldots,T$. When I run a VAR for excess returns of those four indices and realized inflation, as

(6) $$\begin{bmatrix} x_{1,t} \\ \cdot \\ x_{4,t} \\ \pi_t \end{bmatrix} = \begin{bmatrix} c_1 \\ \cdot \\ c_4 \\ c_5 \end{bmatrix} + \begin{bmatrix} \nu_{1,1}(L) & . & . & \nu_{1,5}(L) \\ . & . & . & . \\ \nu_{4,1}(L) & . & . & \nu_{4,5}(L) \\ \nu_{5,1}(L) & . & . & \nu_{5,5}(L) \end{bmatrix} \begin{bmatrix} x_{1,t} \\ \cdot \\ x_{4,t} \\ \pi_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \cdot \\ \varepsilon_{4,t} \\ \varepsilon_{\pi,t} \end{bmatrix},$$

I obtain { $\hat{c}_i$ }, { $\hat{\varepsilon}_{i,t}$ } and { $\hat{\nu}_{i,k}(L)$ }, where

(7) $$\hat{\nu}_{i,k}(L) = \hat{\delta}^1_{i,k} L^1 + \hat{\delta}^2_{i,k} L^2 + \ldots$$

Then, I compute the vector of conditional expected values of excess returns for time *T+1* and expected inflation for time *T+1* as:

(8) $$\begin{bmatrix} E_T x_{1,T+1} \\ \cdot \\ E_T x_{4,T+1} \\ E_T \pi_{T+1} \end{bmatrix} = \begin{bmatrix} \hat{c}_1 \\ \cdot \\ \hat{c}_4 \\ \hat{c}_5 \end{bmatrix} + \begin{bmatrix} \hat{\nu}_{1,1}(L) & . & . & \hat{\nu}_{1,5}(L) \\ . & . & . & . \\ \hat{\nu}_{4,1}(L) & . & . & \hat{\nu}_{4,5}(L) \\ \hat{\nu}_{5,1}(L) & . & . & \hat{\nu}_{5,5}(L) \end{bmatrix} \begin{bmatrix} x_{1,T+1} \\ \cdot \\ x_{4,T+1} \\ \pi_{T+1} \end{bmatrix}.$$

Next, the expected real return on index *i* in period *T+1*, the portfolio formation period, is

(9) $$\begin{bmatrix} E_T r_{1,T+1} \\ \cdot \\ E_T r_{4,T+1} \end{bmatrix} = \begin{bmatrix} E_T x_{1,T+1} \\ \cdot \\ E_T x_{4,T+1} \end{bmatrix} + \begin{bmatrix} r^n_{TB,T+1} \\ \cdot \\ r^n_{TB,T+1} \end{bmatrix} - \begin{bmatrix} E_T \pi_{T+1} \\ \cdot \\ E_T \pi_{T+1} \end{bmatrix}$$

where $r^n_{TB,T+1}$ is the ex ante observed nominal return on Treasury bills for time *T+1*. The expected real return on Treasury bills for time *T+1* is

(10) $$E_T r_{TB,T+1} = r^n_{TB,T+1} - E_T \pi_{T+1}.$$

Finally, the conditional probability distribution for real returns for time *T+1* is determined by

(11) $$\begin{bmatrix} \tilde{r}_{1,T+1} \\ \cdot \\ \tilde{r}_{4,T+1} \\ \tilde{r}_{TB,T+1} \end{bmatrix} = \begin{bmatrix} E_T x_{1,T+1} \\ \cdot \\ E_T x_{4,T+1} \\ 0 \end{bmatrix} + \begin{bmatrix} r^n_{TB,T+1} \\ \cdot \\ r^n_{TB,T+1} \\ r^n_{TB,T+1} \end{bmatrix} - \begin{bmatrix} E_T \pi_{T+1} \\ \cdot \\ E_T \pi_{T+1} \\ E_T \pi_{T+1} \end{bmatrix} + \begin{bmatrix} \tilde{\varepsilon}_{1,T+1} \\ \cdot \\ \tilde{\varepsilon}_{4,T+1} \\ 0 \end{bmatrix} - \begin{bmatrix} \tilde{\varepsilon}_{\pi,T+1} \\ \cdot \\ \tilde{\varepsilon}_{\pi,T+1} \\ \tilde{\varepsilon}_{\pi,T+1} \end{bmatrix}$$

where $\begin{bmatrix} \tilde{\varepsilon}_{1,T+1} \\ . \\ \tilde{\varepsilon}_{4,T+1} \\ \tilde{\varepsilon}_{\pi,T+1} \end{bmatrix}$ takes on the historically observed values $\begin{bmatrix} \tilde{\varepsilon}_{1,t} \\ . \\ \tilde{\varepsilon}_{4,t} \\ \tilde{\varepsilon}_{\pi,t} \end{bmatrix}$ from regression (6), $t=1,2,\ldots,T,$ with equal probabilities ($1/T$).

This way of deriving asset returns probability distribution functions, using historically occurring innovations to asset returns captured through this VAR procedure, is superior to the VAR method mentioned in the literature, e.g. Campbell and Viceira (2002). The literature on derivation of asset returns probability distribution functions assumes that the distribution of asset returns is static, not evolving over time. But the reality is such that the asset returns distribution is dynamic, depending on both recent realizations and the fixed historical distribution of shocks to the dynamic asset returns process. So the right way of deriving asset returns probability distribution functions is to include the dynamics of the past history of asset returns.

Similarly, to get the probability distribution of returns for use in *one-type*-asset portfolios (bonds only or stocks only), the above procedure including the VAR is redone using (6)-(11) with four changed to two.

2.3. Constrained Portfolios

Using the information about the four indices and Treasury bills' derived probability distribution of real returns, I compute constrained optimal portfolios with (a) the two stock indices and Treasury bills, and (b) with the two bond indices and Treasury bills: the solutions of

(12) $$\underset{\{\alpha_1,\alpha_2\}}{Max}\, EU(\tilde{w}) = Max\ E\left\{\frac{1}{\gamma}\left[w_0\left(\alpha_1\tilde{r}_{S\&P500} + \alpha_2\tilde{r}_{NASDAQ} + (1-\alpha_1-\alpha_2)\ \tilde{r}_{TB}\right)\right]^{\gamma}\right\}$$

(13) $$\underset{\{\beta_1,\beta_2\}}{Max}\, EU(\tilde{w}) = Max\ E\left\{\frac{1}{\gamma}\left[w_0\left(\beta_1\tilde{r}_{Corporate\ Bonds} + \beta_2\tilde{r}_{Government\ Bonds} + (1-\beta_1-\beta_2)\ \tilde{r}_{TB}\right)\right]^{\gamma}\right\}$$

where $\alpha_1, \alpha_2, 1\text{-}\alpha_1\text{-}\alpha_2$ and $\beta_1, \beta_2, 1\text{-}\beta_1\text{-}\beta_2$ are the individual portfolio shares in the constrained optimal portfolios with stocks only and bonds only. To get these portfolios I search over α_1, α_2 and β_1, β_2 spaces to optimize expected utility, using nonlinear optimization by a quasi-Newton method based on convergence to first-order conditions of problem (12) and (13). The expectations are taken over the joint probability distributions derived from the *3*-asset VAR.

2.4. Unconstrained portfolios

The next step, then, is to get the unconstrained optimal portfolio with four indices and Treasury bills: the solution of

(14) $$\underset{\{\alpha_1,\ldots,\alpha_4\}}{Max}\, EU(\tilde{w}) =$$

$$= Max\ E\left\{\frac{1}{\gamma}\left[w_0\left(\kappa_1\tilde{r}_{S\&P500} + \kappa_2\tilde{r}_{NASDAQ} + \kappa_3\tilde{r}_{Corporate\ Bonds} + \kappa_4\tilde{r}_{Government\ Bonds} + (1-\kappa_1-\ldots-\kappa_4)\ \tilde{r}_{TB}\right)\right]^{\gamma}\right\}$$

where $\kappa_1, \ldots, \kappa_4$ are the four individual indices' portfolio shares in the unconstrained optimal portfolio. To get the portfolio I search over $\kappa_1, \ldots, \kappa_4$ space to optimize expected utility, again using nonlinear optimization by a quasi-Newton method based on convergence to first-order conditions of problem (14). The expectation is taken over the joint probability distribution derived from the *5*-asset VAR.

2.5. Calculating Opportunity Cost

Now, when I have the constrained and unconstrained optimal portfolios I calculate the proportionate opportunity cost, θ-*1.0*. For the formulas for θ I need to find E($\tilde{R}^{\gamma}$)$^{\text{unconstrained}}$ (where $\tilde{R}$ is the gross return for the optimal unconstrained portfolio with four indices and Treasury bills), and E($\tilde{R}^{\gamma}_{Bonds}$)$^{\text{constrained}}$ and E($\tilde{R}^{\gamma}_{Stocks}$)$^{\text{constrained}}$ (where $\tilde{R}_{Bonds}$ and $\tilde{R}_{Stocks}$ are the gross returns for the optimal constrained portfolios with either two bonds indices and Treasury bills or with two stock indices and Treasury bills). E($\tilde{R}^{\gamma}$)$^{\text{unconstrained}}$ will be equal to:

$$(15)\ \mathrm{E}(\tilde{R}^{\gamma})^{\text{uncd}} = \frac{1}{T}\sum_{t=1}^{T}\left\{\begin{bmatrix}\kappa_1^* & \kappa_2^* & \kappa_3^* & \kappa_4^* & \kappa_5^*\end{bmatrix}\begin{bmatrix} E_T r_{S\&P500,T+1} + \varepsilon_{S\&P500,t} - \varepsilon_{\pi,t} \\ E_T r_{NASDAQ,T+1} + \varepsilon_{NASDAQ,t} - \varepsilon_{\pi,t} \\ E_T r_{CorporateBonds,T+1} + \varepsilon_{CorporateBonds,t} - \varepsilon_{\pi,t} \\ E_T r_{Government\ Bonds,T+1} + \varepsilon_{Government\ Bonds,t} - \varepsilon_{\pi,t} \\ E_T r_{TB,T+1} + 0 - \varepsilon_{\pi,t} \end{bmatrix}\right\}^{\gamma}$$

where vector of κ_i^* is the vector of optimal shares for unconstrained portfolio (with $\kappa_5 = 1 - \kappa_1 - ... - \kappa_4$); the vectors of $E_T r_{i,T+1} + \varepsilon_{i,t} - \varepsilon_{\pi,t}$ and $E_T r_{TB,T+1} + 0 - \varepsilon_{\pi,t}$ for *t=1,…,T* are the vectors of particular possible values of real returns (conditional on the data set for times *t=1* through *T*) at time *T+1*.

And $E(\tilde{R}^{\gamma})^{constrained}$ will be equal to alternatively:

$$(16)\qquad E(\tilde{R}^{\gamma}_{Stocks})^{constrained} = \frac{1}{T}\sum_{t=1}^{T}\left\{\begin{bmatrix}\alpha_1^* & \alpha_2^* & \alpha_3^*\end{bmatrix}\begin{bmatrix} E_T r_{S\&P500,T+1} + \varepsilon_{S\&P500,t} - \varepsilon_{\pi,t} \\ E_T r_{NASDAQ,T+1} + \varepsilon_{NASDAQ,t} - \varepsilon_{\pi,t} \\ E_T r_{TB,T+1} + 0 - \varepsilon_{\pi,t} \end{bmatrix}\right\}^{\gamma}$$

and

$$(17)\quad E(\tilde{R}^{\gamma}_{Bonds})^{consd} = \frac{1}{T}\sum_{t=1}^{T}\left\{\begin{bmatrix}\beta_1^* & \beta_2^* & \beta_3^*\end{bmatrix}\begin{bmatrix} E_T r_{CorporateBonds,T+1} + \varepsilon_{CorporateBonds,t} - \varepsilon_{\pi,t} \\ E_T r_{Government\ Bonds,T+1} + \varepsilon_{Government\ Bonds,t} - \varepsilon_{\pi,t} \\ E_T r_{TB,T+1} + 0 - \varepsilon_{\pi,t}\end{bmatrix}\right\}^{\gamma}$$

where the vectors of α_i^* and β_i^* are the unit-sum vectors of constrained portfolio shares for the "only stocks" portfolio and for the "only bonds" portfolio.

Finally, having calculated (16)-(17), I use (18) and (19) to get a numerical value for θ:

$$(18)\qquad \theta = \left[\frac{E(\tilde{R}^{\gamma}_{Bonds\ and\ Stocks})^{optimal}}{E(\tilde{R}^{\gamma}_{Bonds})^{optimal}}\right]^{\frac{1}{\gamma}};$$

$$(19)\qquad \theta = \left[\frac{E(\tilde{R}^{\gamma}_{Bonds\ and\ Stocks})^{optimal}}{E(\tilde{R}^{\gamma}_{Stocks})^{optimal}}\right]^{\frac{1}{\gamma}}.$$

The denominator expectations are taken over the distributions implied by the relevant *3*-variable VAR, so I am calculating only the cost of investing in two stock or two bond indices rather than in four indices, and not the cost of using a restricted VAR for the portfolio formation.

And the proportionate opportunity cost, θ-*1.0*, will be calculated for both the "only stocks" restriction and the "only bonds" restriction.

The above exercise is done for each of 11 alternative values of the risk aversion parameter, γ.

3. Results

Table 1 reports the results from calculating the proportionate opportunity cost for 11 different values of relative risk aversion for the two types of constrained portfolios:

Table 1

The proportionate opportunity costs, (θ-*1*), optimal unconstrained portfolio weights and optimal ratios of stock indices to bond indices for various values of relative risk aversion

Relative Risk	Opportunity cost of investing		Optimal portfolio weights[1]		Optimal ratios of
Aversion, (1-γ)	in stocks only	in bonds only	of stocks	of bonds	stocks to bonds
0.7	0.013	0.016	7.232	3.409	2.148
1	0.009	0.014	6.948	2.850	2.439
2	0.006	0.009	3.724	1.486	2.506
3	0.004	0.005	2.577	1.002	2.572
9	0.001	0.002	0.867	0.336	2.580
10	0.001	0.002	0.785	0.304	2.582
11	0.001	0.002	0.714	0.276	2.587
12	0.001	0.002	0.656	0.253	2.593
29	0.000	0.000	0.272	0.103	2.641
30	0.000	0.000	0.266	0.100	2.660
31	0.000	0.000	0.258	0.097	2.660

[1]*These two columns give the sum of shares placed in, respectively, the two stock indices or the two bond indices, by unconstrained investors.*

portfolios that consist of two stock indices and Treasury bills, the "only stocks" portfolios, and portfolios that consist of two bond indices and Treasury bills, the "only bonds" portfolios, based on historically occurring asset returns over the *ten*-year period January 1992 through December 2001.

Of all the values of relative risk aversion examined the lowest proportionate opportunity cost of investing in "only stocks" portfolios, 0.00% (0.000), corresponds to the three highest levels of relative risk aversion of 29, 30, and 31. This means that an investor with the level of risk aversion of 29 and higher being unconstrained will be equally happy as if he was constrained. The highest proportionate opportunity cost, 1.3% (0.013), corresponds to the lowest level of relative risk aversion of 0.7 and will be incurred by investors of that level of risk aversion should they decide to invest in "only stocks" portfolios rather than investing in both bonds and stocks. This means that an investor with the level of risk aversion of 0.7 being unconstrained will be equally happy as if he was constrained to stocks only but had 1.3% more initial wealth.

The lowest proportionate opportunity cost of investing in "only bonds" portfolios, 0.00% (0.000), again corresponds to the three highest levels of relative risk aversion of 29, 30, and 31. In this case, the same as with "only stocks" portfolios, an investor with the level of risk aversion of 29 and higher being unconstrained will be equally happy as if he was constrained and had only bonds his portfolio. The highest proportionate opportunity cost, 1.6% (0.016), corresponds to the lowest level of relative risk aversion of 0.7 and will be incurred by investors of that level of risk aversion should they decide to invest in "only bonds" portfolios rather than in portfolios with both bonds and stocks. This means that an investor with the level of risk aversion of 0.7 being unconstrained will be equally happy as if he was constrained to bonds only but had 1.6% more initial wealth.

Table 1 clearly shows that as level of relative risk aversion increases, both proportionate opportunity costs of investing in "stocks only" and "bonds only" portfolios decrease, given the CRRA utility function (1).

These results suggest that optimal unconstrained portfolios offer high risk-tolerance investors broader, more daring investment opportunities than constrained optimal portfolios, and so the investors will require a premium to give up those investment opportunities.

What differs between the values of the proportionate opportunity costs for the two types of constrained optimal portfolios is the magnitude of the cost. For the low levels of risk aversion, from 0.7 to 3, the values of the proportionate opportunity costs of investing in "only stocks" portfolios are lower than those for constrained optimal portfolios of "only bonds". It is reasonable to assume that high risk-tolerance investors, if they are to invest in either constrained portfolios, will prefer "only stocks" portfolio. This conclusion comes from observing the optimal unconstrained portfolio shares for stocks and bonds in Table 1 for high risk-tolerance investors. These investors follow very aggressive short-sell strategies in terms of Treasury bills by placing large proportions, larger than for bond indices, of their initial wealth in stock indices. Even though the constrained "only stocks" portfolios offer higher risk than constrained "only bonds" portfolio, they also offer higher expected returns than constrained "only bonds" portfolios (see Table 2). Therefore, if high risk-tolerance investors were forced to invest in "only bonds" portfolios the opportunity cost they incur under that strategy will be higher than that under "only stocks" strategy.

What is also interesting is the fact that for investors with risk aversion of nine and higher the values of the proportionate opportunity costs of investing in "only stocks" or in "only bonds" are the same. As investors become less and less risk-tolerant they place bigger and bigger proportions of initial wealth into Treasury bills. So, as risk aversion increases, investors if they were forced to invest in stocks and Treasury bills would place a bigger fraction of their initial wealth into Treasury bills and a smaller fraction of initial wealth into stock indices. The same happens to investors with portfolios of bond indices and Treasury bills. This change in portfolio weights makes stocks-and-Treasury-bills portfolios look like bonds-and-Treasury-bills portfolios: portfolios with a large amount of initial wealth placed into Treasury bills, an equivalent of cash, and with a smaller amount of initial wealth placed into risky assets (either bonds or stocks). Therefore, investors with risk aversion of nine and higher, by choosing either constrained portfolio, will incur the proportionate opportunity cost of the same magnitude.

The last column of Table 1 reports the ratios of optimal unconstrained portfolio shares of stocks to bonds. It is clear from the table that as relative risk aversion increases the optimal ratio of stocks to bonds slightly increases, but virtually stays constant. These results (slight increase in the optimal ratios) support Canner, Mankiw and Weil's (1997) and can be explained by the following fact. As investors become less and less risk tolerant, the portion of initial wealth placed into Treasury bills increases (see Table 2). At the same time portions of initial wealth they place into bonds and stock will decrease too but not at the same rate. Given that returns on Treasury bills and bonds are highly correlated, according to Canner, Mankiw and Weil (1997), highly risk-averse investors will reduce the proportion of their initial wealth they place into bonds at a higher rate than that they place into stocks. Therefore, as risk aversion increases, the optimal ratio of stocks to bonds will increase too. But if we consider the optimal ratios from Table 6 as virtually constant, then these results are consistent with the mutual-fund separation theorem according to which the ratio of stocks to bonds is constant for investors with different levels of risk aversion. But in both cases (the optimal ratios are slightly increasing or virtually constant) these results are not consistent with the popular advice according to which more risk-averse investors should hold a lower ratio of stocks to bonds.

Table 2 reports optimal portfolio shares for unconstrained and constrained portfolio strategies for three different levels of relative risk aversion: low (0.7), medium (11) and high (31).

For optimal constrained and unconstrained portfolios for risk aversion of 0.7 more than 100% of initial wealth, w_0, is held in the nominally risky assets, stock and bond indices, and Treasury bills are held in negative quantities.

As risk aversion increases the proportion of initial wealth held in Treasury bills becomes positive and increases for optimal unconstrained and constrained portfolios, and correspondingly the proportion of initial wealth held in bond indices and stock indices decreases.

The table shows that unconstrained (and constrained) optimal portfolio shares are not similar for different levels of risk aversion. As a matter of fact, optimal unconstrained and constrained portfolios for the low level of relative risk aversion of 0.7 have more extreme quantities (negative as well as positive) of assets than optimal unconstrained and constrained portfolios for the medium level of relative risk aversion of 11 and for the high level of relative risk aversion of 31. Extremely negative quantities of assets for high risk-tolerance investors mean that the investors follow an aggressive short sale strategy.

Also Table 2 shows gross monthly expected returns on unconstrained and constrained optimal portfolios, $E\ (X^{*\prime}\tilde{R})$, for the three levels of relative risk aversion (0.7, 11 and 31). The net expected monthly portfolio return (gross expected monthly portfolio return minus *1.0*, multiplied by *100%*) for risk aversion of 0.7 is very dramatic for the unconstrained optimal portfolio, 5.9%, and large for both constrained optimal portfolios (3.7% for “only stocks” and 2.6% for “only bonds” portfolios). Net expected returns are of small size for risk aversion of 11 and of 31. Such extreme magnitudes of expected portfolio returns for high risk-tolerance investors confirm the previously made

Table 2

Optimal portfolio shares for unconstrained and constrained to include stocks only or bonds only portfolios for different values of relative risk aversion

	Treasury bills	Government Long-Term Bonds	Corporate Bonds	S&P 500	NASDAQ	[E(X*' R)[1]	Certainty Equivalent
Relative Risk Aversion, (1-γ), of 0.7							
Unconstrained portfolios of bonds and stocks	-9.641	2.159	1.250	10.308	-3.076	1.059	1.032
Constrained portfolios of stocks	-9.529	0.000	0.000	3.657	6.872	1.037	1.019
Constrained portfolios of bonds	-2.868	3.354	0.514	0.000	0.000	1.026	1.016
Relative Risk Aversion, (1-γ), of 11							
Unconstrained portfolios of bonds and stocks	0.010	0.183	0.093	0.935	-0.221	1.005	1.002
Constrained portfolios of stocks	0.286	0.000	0.000	0.255	0.459	1.003	1.001
Constrained portfolios of bonds	0.724	0.235	0.041	0.000	0.000	1.001	1.000
Relative Risk Aversion, (1-γ), of 31							
Unconstrained portfolios of bonds and stocks	0.645	0.052	0.047	0.322	-0.064	1.003	0.999
Constrained portfolios of stocks	0.744	0.000	0.000	0.082	0.174	1.001	0.999
Constrained portfolios of bond	0.903	0.081	0.016	0.000	0.000	0.999	0.999

[1] *Monthly gross expected return on portfolios.*

conclusion about very aggressive short sale strategies. These magnitudes represent very leveraged portfolios (unconstrained as well as constrained). For investors with risk aversion of 11 and 31 there is some short selling is going on too (in terms of NASDAQ index and only for the unconstrained portfolios), but not as aggressive as for investors with risk aversion of 0.7. The less aggressive short selling for medium or high risk aversion leads to lower mean return portfolios.

In comparing unconstrained expected portfolio returns and constrained expected portfolio returns for the three levels of risk aversion I find that unconstrained and constrained expected portfolio returns for risk aversion of 11 and of 31 are closer to each other than that for risk aversion of 0.7. This shows that as risk aversion increases the more nearly indifferent an investor is between the unconstrained and constrained portfolio strategies.

It is also interesting to compare expected monthly portfolio returns from Table 2 and the values of the proportionate opportunity cost reported in Table 1. For risk aversion of 0.7 the proportionate opportunity cost of investing in the “only stocks” portfolio reaches 1.3% and the proportionate opportunity cost of investing in the “only bonds” portfolio reaches 1.6%; unconstrained investors have the net expected monthly portfolio return (gross expected portfolio return minus *1.0*, multiplied by *100%*) of 5.9%. For risk aversion of 11 the proportionate opportunity cost of investing in either of two constrained portfolios is 0.1% for stocks and 0.2% for bonds, and the net expected monthly portfolio return for unconstrained investors is 0.5%. For the level of risk aversion of 31 the proportionate opportunity cost of investing in a constrained portfolio is 0.0%.

Table 2 also reports the *certainty equivalents* calculated for the same three levels of relative risk aversion (0.7, 11 and 31). The *certainty equivalent*, (*CE*), is defined by

(20) $$\frac{1}{\gamma} CE^{\gamma} = \frac{1}{\gamma} w_0^{\gamma} E\left(\tilde{R}^{\gamma}\right)$$

and so, with $w_0=1$,

(21) $$CE = \left(E\left[\tilde{R}^{\gamma}\right]\right)^{\frac{1}{\gamma}}.$$

The certainty equivalent represents the amount of certain wealth that would be viewed with indifference to the optimal portfolio. It is computed for investors of different levels of risk aversion: low (of 0.7), medium (of 11) and high (of 31). The table shows that as risk aversion increases the value of the certainty equivalent decreases (for the unconstrained portfolio strategy as well as for the constrained). This suggests that as investors become more afraid of risk they use less risky portfolio strategies and will be expecting lower returns from those portfolios and, so, the certain amount of wealth they will be willing to accept with indifference will decrease.

It is interesting to compare certainty equivalents for unconstrained optimal portfolio strategies from Table 2 and the values of the proportionate opportunity cost from Table 1. For the level of risk aversion of 0.7 the proportionate opportunity cost of investing in the "only stocks" portfolio is 1.3% while the unconstrained net certainty equivalent (certainty equivalent minus *1.0*) reaches 3.2%; the proportionate opportunity cost of investing in the "only bonds" portfolio is 1.6%. As the level of risk aversion increases to 31 the proportionate opportunity cost for both types constrained portfolios falls to 0.0% and the unconstrained net certainty equivalent falls to 0.0%.

Table 3 presents the percentage of gross certainty equivalent for unconstrained portfolio strategies lost due to the constraint of investing in “only stocks” portfolios and to the constraint of investing in “only bonds” portfolios, computed as shown in (22). The percentage of gross certainty equivalent lost due to the constraints is timeless just like the proportionate opportunity cost, θ*-1.0*.

$$\text{(22)} \qquad The\ percentage\ loss = \frac{CE^{Uncd} - CE^{Const}}{CE^{Uncd}} * 100\% = \frac{\theta - 1.0}{\theta} * 100\%$$

The highest percentage loss, 1.6%, happens for the investors with risk aversion of 0.7 holding “only bonds” in their portfolios. As risk aversion increases, for both “only stocks” and “only bonds” portfolios, the percentage loss decreases. The lowest percentage loss, 0.0%, is observed for the investors with risk aversion of 29 and higher holding either “only stocks” or “only bonds” in their portfolios. Table 3 confirms the previously made conclusion that as investors become more afraid of risk their perceptions of the optimal constrained and optimal unconstrained portfolio strategies become more and more similar due to investing a large portion of their initial wealth into Treasury bills.

Table 3

The percentage of certainty equivalent lost due to the “only stocks” and “only bonds” constraints for various levels of relative risk aversion

Type of assets	**Relative Risk Aversion, (*1-γ*)**										
	0.7	1	2	3	9	10	11	12	29	30	31
Stocks only	1.3%	0.9%	0.6%	0.4%	0.1%	0.1%	0.1%	0.1%	0.0%	0.0%	0.0%
Bonds only	1.6%	1.4%	0.9%	0.5%	0.2%	0.2%	0.2%	0.1%	0.0%	0.0%	0.0%

$$The\ percentage\ loss = \frac{CE^{Uncd} - CE^{Cond}}{CE^{Uncd}} * 100\% = \frac{\theta - 1}{\theta} * 100\%.$$

Hence, as the level of risk aversion increases the certain amount of wealth unconstrained investors and constrained investors will be willing to accept with indifference will be getting closer to each other, and, therefore, the percentage loss in certainty equivalents due to the constraint will decrease.

4. Conclusion

In this paper I have investigated the opportunity cost incurred by investors when they are constrained to use only one type of assets, either stocks or bonds, instead of being unconstrained and using both stocks and bonds. The original historical asset returns are used along with CRRA utility functions, and the proportionate opportunity cost. The opportunity cost has been calculated for different values of relative risk aversion (including extreme levels of relative risk aversion) for “only stocks” and “only bonds” portfolios. The highest proportionate opportunity cost found is 1.6% (0.016) for the level of relative risk aversion of 0.7 for “only bonds” portfolios. The lowest proportionate opportunity cost found is 0.0% (0.000) for the level of relative risk aversion of 29 and higher for both types of constrained portfolio strategies. I found for both types of constrained optimal portfolios that as the level of relative risk aversion increases the proportionate opportunity cost decreases.

The only difference between estimates of the proportionate opportunity cost for the two constrained portfolios is the magnitude of the estimates. They are bigger for the level of risk aversion of 0.7 for the “only bonds” portfolios. This can be explained by the fact that high risk-tolerance investors prefer “only stocks” portfolios to “only bonds” portfolios and will be more satisfied with “only stocks” portfolios that offer higher risk and higher returns than with “only bonds” portfolios. Therefore, the investors will require a lower proportion of initial wealth while being constrained by “only stocks” than by “only bonds”. However, the difference in these opportunity costs is slight.

My findings of optimal ratios of stocks to bonds for different levels of risk aversion confirm the mutual-fund separation theorem and contradict popular financial advice. I found that as the level of risk aversion increases the optimal ratio of stocks to bonds virtually stays constant.

Based on my calculations, I may conclude that for investors with high levels of relative risk aversion (nine and above), constrained portfolios that include only one type of assets, stocks only or bonds only, along with Treasury bills perform as well as unconstrained portfolios that include both types of assets, stocks and bonds.

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