The Opportunity Cost For An Investor Of Being Constrained By The Mean-Variance Framework

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Abstract

Mean-variance analysis as a constrained portfolio strategy gives an investor a sub-optimal asset allocation that results in a welfare loss for the investor. To measure that welfare loss I compare mean-variance constrained efficient portfolios with optimal unconstrained portfolios by using the concept of the proportionate opportunity cost along with various CRRA utility functions. A vector autoregression is used to generate the joint distribution of asset returns for the portfolio formation period. I show that investors' welfare losses do not exceed 5.6% of initial wealth when extreme values of returns are not exaggerated in the returns distribution. With extreme returns values exaggerated, investors' welfare losses do not exceed 11% of initial wealth. For both cases as the number of assets in investors' portfolios increases investors' welfare losses from the mean-variance constraint increase as well, and less risk-averse investors experience greater welfare losses.

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Keywords: Probability distribution function of stock returns; Proportionate opportunity cost; Mean-variance efficient portfolio strategy; Optimal portfolio strategy; Investors' welfare losses

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1. Introduction

Mean-variance analysis is used very frequently as a sub-optimal constrained portfolio strategy. Any constrained portfolio strategy will give an investor a sub-optimal asset allocation and that will result in the investor's experiencing welfare loss. How large can that welfare loss for the investor be if he has chosen the mean-variance efficient portfolio instead of the optimal portfolio? In order to answer this question I will compare expected utility from the sub-optimal asset allocation (the mean-variance efficient constrained portfolio) with that from optimal asset allocation by using the concept of opportunity cost.

The best way to measure investors' welfare losses is to use the proportionate opportunity cost. It is natural because the results are readily interpretable as intuitively "large" or "small". I assume the constant relative risk aversion (CRRA) utility function, because this function is commonly used and it is plausible (e.g. has decreasing absolute risk aversion (DARA) preferences) and mathematically tractable when combined with the proportionate opportunity cost. Thus

(1) $$U(\tilde{w}) = \begin{cases} \frac{1}{\gamma}\tilde{w}^{\gamma}, \gamma < 1, \gamma \neq 0, \tilde{w} > 0 \\ -\infty, \tilde{w} \leq 0 \end{cases}$$

under this utility function the proportionate opportunity cost (willingness to accept payment as compensation for being constrained) can be calculated as θ - *1.0* where θ is defined by

(2) $$EU(\theta\, w_0\, \tilde{R}^c) = EU(w_0\, \tilde{R}^u)$$

where w_0 is the initial wealth, and $\tilde{R}^u$ and $\tilde{R}^c$ are the stochastic returns per dollar invested for the unconstrained and constrained portfolios. Solving (2) gives

(3) $$\theta = \left[\frac{E(\tilde{R}^{\gamma})^{unconstrained}}{E(\tilde{R}^{\gamma})^{constrained}} \right]^{\frac{1}{\gamma}}.$$

The literature contains two examples of measuring the opportunity cost of using the mean-variance efficient portfolio instead of the optimal portfolio that I would like to discuss first.

Simaan (1993) used historical stock returns and their joint distribution was specified by a joint distribution, with a normal distribution of the idiosyncratic shocks conditional on a single common factor with the Pearson Type III class distribution. For a single period portfolio selection problem he derived closed-form solutions for the optimal portfolio under constant absolute risk aversion (CARA) utility function, and for the investor's second best choice, the best mean-variance efficient portfolio. Then he computed empirically the size of the optimization premium, ρ, for replacing the investor's second best choice by the optimal portfolio. This optimization premium, ρ, was expressed additively in dollar terms, Simaan (1993, p. 579): "miminum amount on an invested dollar that an investor would require in order to replace his optimal strategy with his best mean-variance investment strategy". Thus Simaan's compensation ρ is added to final wealth. In contrast, what I am going to do is to work with the proportionate opportunity cost, θ*-1.0*: that is, an optimization premium that is expressed as a fraction of initial wealth. If I were to construct in Simaan's context the analog of the proportionate opportunity cost, θ*-1.0*, using his definition for the additive ρ, then the analog would equal $\frac{\rho}{w_0}$ and would depend on initial wealth (since his ρ does not due to his use of CARA utility). The proportionate optimization premium expressed that way is not appealing because it can take on any value, large or small, depending on the level of initial wealth: for investors with lower initial wealth the proportionate optimization premium will be higher than for investors with high initial wealth. What we need is a way to express the proportionate optimization premium so it does not depend on initial wealth; the proportionate opportunity cost, θ-*1.0*, along with CRRA, instead of CARA, is the way to do that.

Tew, Reid and Witt (1991) used a Monte Carlo approach to simulate asset returns distributions using various assumptions about random characteristics of hypothetical investments. Several parameterizations of CARA and CRRA utility functions were used, including some cases of extreme risk aversion characteristics. Simulated data sets of asset returns and these utility functions were used to compute the opportunity cost of accepting the mean-variance investment strategy. By using six different utility functions Tew, Reid and Witt wanted to illustrate the limits of the mean-variance approximation of optimal investment strategy. And that is why they chose extreme risk aversion characteristics. Looking for the limits of the mean-variance investment strategy the authors employed the concept of additive opportunity cost in dollar terms used by Simaan. And as I have mentioned before, the concept is not appealing. The best way to approach the problem of computing the opportunity cost, which I will follow here, is to use the proportionate opportunity cost along with the CRRA utility function (1). I will use different levels of risk aversion to illustrate the limits of the mean-variance investment strategy.

My way of deriving joint probability distributions of asset returns is different from that of Simaan and Tew, Reid and Witt. Simaan assumed a particular class of joint distributions fitted to historical data, and Tew, Reid and Witt simulated data from assumed security returns distributions. In both cases optimized portfolios were based on those assumed distributions. I am going to use a vector autoregressive process (VAR) to project the means of returns and to capture 120 historically occurring shocks to all asset returns, and then I will assume that the true distribution of shocks for the investment period is given by those 120 sets of returns shocks with equal probabilities. I do not fit a continuous distribution to the data because the results may be sensitive to the assumed distribution. The present way of deriving joint probability distributions provides the most defensible representation of the current asset returns distributions facing investors in the portfolio formation period.

The procedure of calculating the proportionate opportunity cost for an investor of being constrained by the mean-variance framework includes random asset selection for investors' portfolios, an estimation of a vector autoregressive process, derivation of the joint probability distribution function of asset returns, and computing mean-variance efficient constrained optimal and unconstrained optimal portfolios.

In this paper I show that with a nominally risk-free asset, as relative risk aversion increases the mean-variance strategy shows a moderately good approximation to the optimal portfolio strategy. The results show that investors' welfare losses do not exceed 5.6% of initial wealth. The results also show investors' welfare losses become larger when the number of assets in portfolios increases.

The second section of this paper describes the procedure of random asset selection for investors' portfolios, of inferring the joint probability distribution function of asset returns, of computing the mean-variance efficient constrained optimal and unconstrained optimal portfolios, and the calculation of the proportionate opportunity cost. The third section discusses the results of the study, and the fourth section concludes.

2. The Procedure

2.1. Asset selection

The procedure of calculating the proportionate opportunity cost for different levels of risk aversion will be performed 1,000 times, in each case using 25 randomly picked nominally risky assets and Treasury bills as the nominally risk-free asset. Then, the entire procedure will be repeated for eight randomly picked nominally risky assets and Treasury bills.

The first step is to pick at random 25 nominally risky assets. Then, to construct the optimal constrained and optimal unconstrained portfolios I need to get expected values of real returns for the 26 assets I am using for time $T+1$: for the 25 nominally risky assets and for nominally risk-free Treasury bills. In real terms, though, there is no risk-free asset. Returns on Treasury bills are risk-free only in nominal terms. But in time-series data inflation will be uncertain in any period and, thus, so will the real rate of return on Treasury bills. Therefore, the 26 assets that I am dealing with in real terms will all be risky assets. The same procedures are also conducted with nine assets instead of 26.

A set of nine or 26 assets is big enough to give reliable estimates of optimal constrained and optimal unconstrained portfolios and, therefore, for the opportunity cost. Simaan (1993) argued that ten stocks will be sufficient to trace the efficient frontier and Tew, Reid and Witt (1991) used from two to nine stocks in their calculations.

2.2. Vector autoregressions of returns

So, to get expected values of real returns for the case of 26 assets at time *T+1*, the portfolio formation period, I estimate a vector autoregressive process (VAR). The next steps are to derive the joint probability distribution for the 26 assets' real returns, and, finally, to construct optimal constrained and optimal unconstrained portfolios.

To derive the joint probability distribution of empirical deviations from the VAR-estimated conditional means for those randomly picked asset returns I do the following.

The nominal return on asset *i* at time *t* minus the nominal return on Treasury bills at time *t* gives us the excess return on asset *i* ($x_{i,t}$) at time *t* for $i=1,\ldots,25$ and for $t=1,\ \ldots,T$. When I run a VAR for excess returns of those 25 assets and realized inflation, as

$$(4)\qquad \begin{bmatrix} x_{1,t} \\ \cdot \\ x_{25,t} \\ \pi_t \end{bmatrix} = \begin{bmatrix} c_1 \\ \cdot \\ c_{25} \\ c_{26} \end{bmatrix} + \begin{bmatrix} \nu_{1,1}(L) & \cdot & \cdot & \nu_{1,26}(L) \\ \cdot & \cdot & \cdot & \cdot \\ \nu_{25,1}(L) & \cdot & \cdot & \nu_{25,26}(L) \\ \nu_{26,1}(L) & \cdot & \cdot & \nu_{26,26}(L) \end{bmatrix} \begin{bmatrix} x_{1,t} \\ \cdot \\ x_{25,t} \\ \pi_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \cdot \\ \varepsilon_{25,t} \\ \varepsilon_{\pi,t} \end{bmatrix},$$

I obtain $\{\hat{c}_i\}$, $\{\hat{\varepsilon}_{i,t}\}$ and $\{\hat{\nu}_{i,k}(L)\}$, where

$$(5)\qquad \hat{\nu}_{i,k}(L) = \hat{\delta}^1_{i,k}L^1 + \hat{\delta}^2_{i,k}L^2 + \ldots$$

Then, I compute the vector of conditional expected values of excess returns for time *T+1* and expected inflation for time *T+1* as:

$$(6)\qquad \begin{bmatrix} E_T x_{1,T+1} \\ \cdot \\ E_T x_{25,T+1} \\ E_T \pi_{T+1} \end{bmatrix} = \begin{bmatrix} \hat{c}_1 \\ \cdot \\ \hat{c}_{25} \\ \hat{c}_{26} \end{bmatrix} + \begin{bmatrix} \hat{\nu}_{1,1}(L) & \cdot & \cdot & \hat{\nu}_{1,26}(L) \\ \cdot & \cdot & \cdot & \cdot \\ \hat{\nu}_{25,1}(L) & \cdot & \cdot & \hat{\nu}_{25,26}(L) \\ \hat{\nu}_{26,1}(L) & \cdot & \cdot & \hat{\nu}_{26,26}(L) \end{bmatrix} \begin{bmatrix} x_{1,T+1} \\ \cdot \\ x_{25,T+1} \\ \pi_{T+1} \end{bmatrix}.$$

Next, the expected real return on asset *i* in period *T+1*, the portfolio formation period, is

(7)
$$\begin{bmatrix} E_T r_{1,T+1} \\ . \\ E_T r_{25,T+1} \end{bmatrix} = \begin{bmatrix} E_T x_{1,T+1} \\ . \\ E_T x_{25,T+1} \end{bmatrix} + \begin{bmatrix} r^n_{TB,T+1} \\ . \\ r^n_{TB,T+1} \end{bmatrix} - \begin{bmatrix} E_T \pi_{T+1} \\ . \\ E_T \pi_{T+1} \end{bmatrix}$$

where $r^n_{TB,T+1}$ is the ex ante observed nominal return on Treasury bills for time *T+1*. The expected real return on Treasury bills for time *T+1* is

(8)
$$E_T r_{TB,T+1} = r^n_{TB,T+1} - E_T \pi_{T+1}.$$

Finally, the conditional probability distribution for real returns for time T+1 is determined by

(9)
$$\begin{bmatrix} \tilde{r}_{1,T+1} \\ . \\ \tilde{r}_{25,T+1} \\ \tilde{r}_{TB,T+1} \end{bmatrix} = \begin{bmatrix} E_T x_{1,T+1} \\ . \\ E_T x_{25,T+1} \\ 0 \end{bmatrix} + \begin{bmatrix} r^n_{TB,T+1} \\ . \\ r^n_{TB,T+1} \\ r^n_{TB,T+1} \end{bmatrix} - \begin{bmatrix} E_T \pi_{T+1} \\ . \\ E_T \pi_{T+1} \\ E_T \pi_{T+1} \end{bmatrix} + \begin{bmatrix} \tilde{\varepsilon}_{1,T+1} \\ . \\ \tilde{\varepsilon}_{25,T+1} \\ 0 \end{bmatrix} - \begin{bmatrix} \tilde{\varepsilon}_{\pi,T+1} \\ . \\ \tilde{\varepsilon}_{\pi,T+1} \\ \tilde{\varepsilon}_{\pi,T+1} \end{bmatrix}$$

where $\begin{bmatrix} \tilde{\varepsilon}_{1,T+1} \\ . \\ \tilde{\varepsilon}_{25,T+1} \\ \tilde{\varepsilon}_{\pi,T+1} \end{bmatrix}$ takes on the historically observed values $\begin{bmatrix} \tilde{\varepsilon}_{1,t} \\ . \\ \tilde{\varepsilon}_{25,t} \\ \tilde{\varepsilon}_{\pi,t} \end{bmatrix}$ from equation (4), *t=1,2,…,T,* with equal probabilities (*1/T*).

This way of deriving asset returns probability distribution functions, using historically occurring innovations to asset returns captured through the VAR procedure, is superior to the method mentioned in the literature (e.g. Campbell and Viceira, 2002). The literature on derivation of asset returns probability distribution functions assumes that the distribution of asset returns is static, not evolving over time. But the reality is such that the asset returns distribution is dynamic, depending on both recent realizations and the fixed historical distribution of shocks to the dynamic asset returns process. So the right way of deriving asset returns probability distribution functions is to include the dynamics of the past history of asset returns.

2.3. Mean-variance efficient constrained portfolios

Using the information about those randomly picked assets' derived probability distributions for their real returns (computed as shown in (9)), I compute two mean-variance efficient mutual funds, X_1^* and X_2^*, using the following formula (Merton (1972)):

(10) $X_i^* = \frac{w_0}{\Delta}\left[\left(\bar{r}'V^{-1}\bar{r}\right)V^{-1}k - \left(k'V^{-1}\bar{r}\right)V^{-1}\bar{r}\right] + \frac{\mu_i}{\Delta}\left[\left(k'V^{-1}k\right)V^{-1}\bar{r} - \left(\bar{r}'V^{-1}k\right)V^{-1}k\right]$, *i=1,2*

where

(11) $$\Delta = \left(\bar{r}'V^{-1}\bar{r}\right)\left(k'V^{-1}k\right) - \left(\bar{r}'V^{-1}k\right)^2 > 0.$$

Here w_0 is the initial wealth that is set equal to *1*, $\bar{r}$ is a column vector with dimension of *26x1* of expected values of the gross real returns, $\tilde{r}_t$, on 25 nominally risky assets for time *T+1* and expected real return on Treasury bills for time *T+1* calculated as shown in (7) and (8), X_i^* is a mean-variance efficient mutual fund (a column vector of portfolio shares for the 25 picked assets and Treasury bills) with dimension *26x1*, μ_i is expected gross portfolio return (for two different mean-variance efficient mutual funds I pick two different arbitrary values for μ), V is the covariance matrix of real returns with dimension of *26x26* for the 26 risky assets calculated from the distribution of $\begin{bmatrix} \tilde{\varepsilon}_{1,T+1} \\ \cdot \\ \tilde{\varepsilon}_{25,T+1} \\ 0 \end{bmatrix} - \begin{bmatrix} \tilde{\varepsilon}_{\pi,T+1} \\ \cdot \\ \tilde{\varepsilon}_{\pi,T+1} \\ \tilde{\varepsilon}_{\pi,T+1} \end{bmatrix}$

stated after (9), and k is the column vector of 1's with dimension *26x1*.

Computing two mean-variance efficient mutual funds gives us two column vectors, X_1^* and X_2^*, of portfolio shares for the 26 assets for the two mean return levels, μ_1 and μ_2.

The next step is to optimize the expected value of a CRRA utility function, (1), subject to the constraint of being mean-variance efficient, with respect to the single choice variable β: how much to hold in one of the efficient mutual funds as opposed to the other:

(12) $$\underset{\{\beta\}}{Max}\ \mathrm{EU}(\tilde{w}) = \mathrm{E}\{\frac{1}{\gamma}\tilde{w}^{\gamma}\} \text{ subject to } \tilde{w} = [\beta\,(X_1^{*\prime}\tilde{r}_t) + (1-\beta)(X_2^{*\prime}\tilde{r}_t)]w_0$$

where w_0 is the initial wealth that is set equal to *1*, the expectation is taken over the joint probability distribution derived as described above in (4)-(9), and the *time* subscripts on E_T and $\tilde{w}_{T+1}$ have been suppressed for convenience.

By solving the above problem (finding the hilltop in the graph with the Expected utility on the vertical axis and β (portfolio share of mutual fund number one) on the horizontal axis) I get the mean-variance efficient constrained optimal portfolio.

2.4. Unconstrained portfolios

The next step, then, is to get the unconstrained optimal portfolio: the solution of

(13) $$\underset{\{\alpha_1,\ldots,\alpha_{25}\}}{Max} EU(\tilde{w}) = Max\ E\left\{\frac{1}{\gamma}\left[w_0\left(\alpha_1\tilde{r}_1 + \ldots + \alpha_{25}\tilde{r}_{25} + \left(1-\alpha_1-\ldots-\alpha_{25}\right)\tilde{r}_{TB}\right)\right]^{\gamma}\right\}$$

where $\alpha_1, \ldots, \alpha_{25}$ are the first 25 individual assets portfolio shares in the unconstrained optimal portfolio. To get the portfolio I search over $\alpha_1, \ldots, \alpha_{25}$ space to optimize expected utility, using nonlinear optimization by a quasi-Newton method based on convergence to first-order conditions of problem (13). Again, the expectation is taken over the joint probability distribution derived as described above in (4)-(9).

2.5. *Calculating the proportionate opportunity cost*

Now, when I have the constrained optimal and unconstrained optimal portfolios I calculate the opportunity cost, θ-*1.0*. For the formula for θ, (3), I need to find E($\tilde{R}^{\gamma}$)$^{\text{unconstrained}}$ and E($\tilde{R}^{\gamma}$)$^{\text{constrained}}$.

E($\tilde{R}^{\gamma}$)$^{\text{unconstrained}}$ (referring more completely to $E_T(\tilde{R}^{\gamma}_{T+1})^{unconstrained}$) is equal to

$$(14)\quad E_T(\tilde{R}^{\gamma}_{T+1})^{unconstrd} = \frac{1}{T}\sum_{t=1}^{T}\left\{\left[\alpha_1^* \quad \ldots \quad \alpha_{25}^* \quad 1-\alpha_1^*-\ldots-\alpha_{25}^*\right]\begin{bmatrix} E_T r_{1,T+1}+\varepsilon_{1,t}-\varepsilon_{\pi,t} \\ . \\ E_T r_{25,T+1}+\varepsilon_{25,t}-\varepsilon_{\pi,t} \\ E_T r_{TB,T+1}+\ 0-\varepsilon_{\pi,t}\end{bmatrix}\right\}^{\gamma}$$

where the vector of α_i^* is the vector of optimal portfolio shares; the vector of $E_T r_{i,T+1}+\varepsilon_{i,t}$ - $\varepsilon_{\pi,t}$ and $E_T\, r_{TB,T+1}$ - $\varepsilon_{\pi,t}$ is the vector of particular possible values of real returns (conditional on data set for times *t=1* through *T*) at time *T+1* (the portfolio formation period) and calculated as shown in (4)-(9).

And E($\tilde{R}^{\gamma}$)$^{\text{constrained}}$ is equal to

$$(15)\qquad E_T(\tilde{R}^{\gamma}_{T+1})^{constrained} = \frac{1}{T}\sum_{t=1}^{T}\left\{\left[\beta^* X_1^{*\prime} + (1-\beta^*)X_2^{*\prime}\right]\begin{bmatrix} E_T r_{1,T+1}+\varepsilon_{1,t}-\varepsilon_{\pi,t} \\ . \\ E_T r_{25,T+1}+\varepsilon_{25,t}-\varepsilon_{\pi,t} \\ E_T r_{TB,T+1}+\ 0-\varepsilon_{\pi,t}\end{bmatrix}\right\}^{\gamma}$$

where β^* is the portfolio share of mutual fund number one and $(1-\beta^*)$ is the portfolio share of mutual fund number two; X_i^*, for $i=1,2$, are two mean-variance efficient mutual funds.

Then, having calculated (14) and (15), I use (3) to get a numerical value for θ.

The whole procedure, starting from picking 25 (or eight) nominally risky assets, is being repeated 1,000 times. This gives me 1,000 values of θ. The procedure is done for each of 11 alternative values of the risk aversion parameter γ.

3. Results

The results from this research project are as follows.

3.1. Results derived from historical returns data set with no exaggeration of extreme returns

3.1.1. Opportunity costs

Table 1 and Table 2 represent the results from calculation of 1,000 values of the proportionate opportunity cost for 11 different values of relative risk aversion for alternatively 26 and nine assets, based on historically occurring asset returns over the ten-year period January 1992 through December 2001.

Of all the values of relative risk aversion examined the lowest mean (over 1,000 replications) of the proportionate opportunity cost for both 26 and nine assets corresponds to the high level of relative risk aversion of 31. The highest mean (over 1,000 replications) of the proportionate opportunity cost for both 26 and nine assets corresponds to the lowest level of relative risk aversion of 0.7. This suggests that optimal unconstrained portfolios offer high risk-tolerance investors broader, more daring

Table 1

The proportionate opportunity cost, (θ - *1*), for various values of relative risk aversion for 26 assets

Relative Risk Aversion, (*1*-γ)	*Smallest*	*Mean*	*Median*	*Largest*	*Standard Deviation*
Low					
0.7	0.008	0.056	0.044	0.334	0.030
0.999	0.006	0.048	0.027	0.182	0.032
2	0.002	0.035	0.016	0.125	0.029
3	0.001	0.032	0.014	0.114	0.022
Medium					
9	0.000	0.030	0.001	0.111	0.019
10	0.000	0.028	0.001	0.088	0.019
11	0.000	0.024	0.001	0.075	0.018
12	0.000	0.019	0.001	0.040	0.016
High					
29	0.000	0.012	0.001	0.042	0.012
30	0.000	0.006	0.000	0.041	0.012
31	0.000	0.005	0.000	0.028	0.011

Table 2

The proportionate opportunity cost, (θ - *1*), for various values of relative risk aversion for nine assets

Relative Risk Aversion, (*1*-γ)	*Smallest*	*Mean*	*Median*	*Largest*	*Standard Deviation*
Low					
0.7	0.001	0.035	0.004	0.593	0.104
0.999	0.000	0.028	0.002	0.580	0.058
2	0.000	0.014	0.001	0.571	0.028
3	0.000	0.011	0.001	0.395	0.020
Medium					
9	0.000	0.009	0.006	0.321	0.018
10	0.000	0.007	0.006	0.189	0.019
11	0.000	0.007	0.005	0.093	0.018
12	0.000	0.006	0.005	0.080	0.018
High					
29	0.000	0.005	0.005	0.061	0.017
30	0.000	0.004	0.004	0.055	0.017
31	0.000	0.004	0.003	0.012	0.014

investment opportunities than constrained optimal portfolios, and so the investors will require a premium to give up those investment opportunities.

Both tables clearly show that as level of relative risk aversion increases the proportionate opportunity cost decreases, given the CRRA utility function, (1), the better the mean-variance efficient portfolio performs and the lower the proportion of initial wealth an investor requires to stay constrained and accept the mean-variance efficient portfolio instead of optimal unconstrained portfolio. This is not surprising. As risk aversion decreases, as investor becomes more risk tolerant, he considers optimal unconstrained portfolio as his best choice that does not place any restrictions on his investment behavior and let him follow a very aggressive short sale strategy that will not be possible under the constrained portfolio strategy (see Table 3 and Table 4), and, therefore, he will require higher proportion of initial wealth as the payment to stay constrained and accept the optimal constrained mean-variance efficient portfolio.

These results confirm Simaan's (1993) conclusions in the case where a riskless asset was introduced. He also found that as the level of risk aversion increases the optimization premium will decrease. What differs, though, between his results and mine is the magnitude of the estimates of the proportional opportunity cost.

The highest opportunity cost that Simaan found is 30% for the case with no riskless asset and 0.5% for the case with the riskless asset. The highest opportunity cost I have found is 5.6% for 26 assets and 3.5% for nine assets (in both my cases a nominally risk-free asset, risky in real terms, was introduced). The difference can be explained, first, by the use of the utility function with constant relative risk aversion preferences rather than utility function with constant absolute risk aversion preferences as in Simaan, and second, by the fact that my highest proportionate opportunity cost corresponds to the level of risk aversion of 0.7 whereas Simaan's highest opportunity cost corresponds to the level of risk aversion of two (he deliberately did not include levels of risk aversion less then two arguing that they correspond to very aggressive infeasible investment strategies).

Tew, Reid and Witt (1991), working only with risky assets and relative risk aversion that ranged from 0.1 to 1.9 (and with various CRRA utility functions), found that as risk aversion increases the opportunity cost increases too. Their findings are consistent with Simaan's case where there was no riskless asset introduced. The magnitude of their opportunity cost, though, differs from that of Simaan's. Tew, Reid and Witt also found that as the number of assets in the portfolio increases the opportunity cost decreases. It is hard to comment on that conclusion because even though I did consider portfolios of different sizes (26-asset portfolio and nine-asset portfolio), both my portfolios include a nominally risk-free asset that was absent from any of Tew, Reid and Witt's portfolios.

Table 1 and Table 2 also show that as the level of relative risk aversion increases the standard deviations of the proportionate opportunity costs decrease: the distributions of the opportunity cost are getting "tighter". As the level of relative risk aversion increases more and more of the numerical values for the opportunity costs are concentrating around their means. Thus we see that as the level of risk aversion increases, as investors become less risk tolerant, the perceptions of the optimal constrained mean-variance portfolio strategy for investors with different asset sets are more similar to each other than perceptions of that strategy for different investors with lower risk aversion.

For 26 assets, Table 1 shows that the lowest mean (over 1,000 replications) of the proportionate opportunity cost, 0.5% (0.005), corresponds to the high level or risk aversion of 31. This means that an investor with the level of relative risk aversion of 31 being unconstrained will be equally happy as if he was constrained but had 0.5% more of initial wealth. The highest mean (over 1,000 replications) of the proportionate opportunity cost, 5.6% (0.056), corresponds to the very low level of relative risk aversion of 0.7. This means that an investor with the level of relative risk aversion of 0.7 being unconstrained will be equally happy as if he was constrained but had 5.6% more of initial wealth.

For low levels (from three to 0.7) of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost ranges from 3.2% (0.032) for relative risk aversion of three to 5.6% (0.056) for relative risk aversion of 0.7.

For medium (from 12 to nine) levels of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost ranges from 1.9% (0.019) for relative risk aversion of 12 to 3.0% (0.030) for relative risk aversion of nine. This suggests that even medium risk-tolerance investors value optimal unconstrained portfolios high enough as oppose to constrained investment behavior to require from 1.9% to 3.0% of additional initial wealth to stay constrained.

For nine assets, Table 2 shows that the lowest mean (over 1,000 replications) of the proportionate opportunity cost, 0.4% (0.004), corresponds to the high level of risk aversion of 31. This means that an investor with the level of relative risk aversion of 31 being unconstrained will be equally happy as if he was constrained but had 0.4% more of initial wealth. The highest mean (over 1,000 replications) of the proportionate opportunity cost, 3.5% (0.035), corresponds to the very low level of relative risk aversion of 0.7. This means that an investor with the level of relative risk aversion of 0.7 being unconstrained will be equally happy as if he was constrained but had 3.5% more of initial wealth.

The highest values of the proportionate opportunity cost (the means over 1,000 replications) correspond to low levels of relative risk aversion (from 0.7 to three) and range from 1.1% (0.011) for relative risk aversion of three to 3.5% (0.035) for relative risk aversion of 0.7. Investors with low levels of risk aversion (from three to 0.7) in the presence of nine assets will require from 1.1% to 3.5% of initial wealth to stay constrained and accept the mean-variance constrained optimal portfolio. These magnitudes, as a matter of fact, are lower than those for 26 assets in Table 1. This suggests that in the presence of a greater number of available assets investors find broader and more daring investment strategies that are further away from the mean-variance efficient one, and correspond to a higher proportionate opportunity cost.

For medium (from 12 to nine) levels of relative risk aversion the mean (over 1,000 replications) of the proportionate opportunity cost ranges from 0.6% (0.006) for relative risk aversion of 12 to 0.9% (0.009) for relative risk aversion of nine. These numbers are about three times as small as those for 26 assets.

These two tables suggest that the more assets are available for investors the further away low risk aversion investors will go from the mean-variance constrained strategy.

3.1.2. Optimal portfolio shares

Table 3 and Table 4 present typical optimal portfolio shares for unconstrained and constrained portfolio strategies for three different levels of relative risk aversion: low (of 0.7), medium (of 11) and high (of 31), for 26 assets and for nine assets, in each case for a different set of available assets giving an opportunity cost that is typical for that level of risk aversion.

For both tables for risk aversion of 0.7 more than 100% of initial wealth, w_0, is held in the nominally risky assets (asset #1 through asset #25 in Table 3 and asset #1 through asset #8 in Table 4) as a group, and Treasury bills are held in negative quantities.

As risk aversion increases, as investors become more conservative and less risk-tolerant, the proportion of initial wealth held in Treasury bills increases, and correspondingly the proportion of initial wealth held in the group of nominally risky assets decreases.

The tables show that unconstrained (and constrained) optimal portfolio shares are not similar for different levels of risk aversion. As a matter of fact, optimal unconstrained and constrained portfolios for the low level of relative risk aversion of 0.7 have more extreme quantities (negative as well as positive) of assets than optimal unconstrained and constrained portfolios for medium level of relative risk aversion of 11 and for high level of relative risk aversion of 31. Extremely negative quantities of assets for high risk-tolerance investors mean that the investors follow an aggressive short sale strategy.

The extremely negative quantities of assets in portfolios for low risk aversion of 0.7 confirm Simaan (1993), who found exactly the same thing: investors with low levels of risk aversion follow very aggressive short sale strategies.

Table 3

Illustrative optimal portfolio shares for unconstrained and mean-variance constrained portfolio strategies for different values of relative risk aversion for 26 assets[1]

# of An Asset	Relative Risk Aversion,(1-γ), equal to 0.7		Relative Risk Aversion, (1-γ), equal to 11		Relative Risk Aversion, (1-γ), equal to 31	
	Unconstrained	Constrained	Unconstrained	Constrained	Unconstrained	Constrained
1	0.679	0.381	-0.022	-0.022	-0.002	0.001
2	2.168	1.635	0.076	0.065	0.000	-0.006
3	0.525	0.445	0.016	0.037	0.021	0.017
4	2.884	3.649	0.283	0.289	0.003	0.006
5	0.239	0.327	0.043	0.042	0.179	0.148
6	1.022	0.808	0.053	0.062	0.056	0.076
7	0.360	0.470	-0.033	-0.086	-0.018	-0.013
8	-2.332	-2.483	0.121	0.112	0.033	0.025
9	-1.082	-0.795	0.050	0.068	0.094	0.079
10	4.884	2.855	0.046	0.060	-0.046	-0.019
11	0.363	0.533	-0.012	-0.014	-0.119	-0.100
12	-4.082	-2.547	-0.005	-0.061	0.012	0.015
13	4.073	6.576	-0.006	-0.016	-0.058	-0.074
14	-0.419	-0.995	-0.035	-0.061	-0.025	-0.018
15	-0.879	-1.147	0.078	0.219	0.324	0.292
16	1.216	1.457	-0.118	-0.094	0.035	0.026
17	4.971	5.163	0.151	0.126	0.000	0.000
18	3.195	3.513	0.090	0.130	-0.127	-0.085
19	2.554	0.649	-0.154	-0.147	-0.023	-0.018
20	-3.750	-1.816	-0.031	-0.013	-0.005	-0.011
21	-0.014	0.129	-0.158	-0.175	0.000	-0.001
22	-0.728	0.154	-0.027	-0.026	0.041	0.040
23	-1.793	-2.595	0.035	0.036	0.046	0.032
24	1.913	0.926	0.452	0.332	0.021	0.027
25	-0.070	0.333	-0.111	-0.089	0.004	-0.005
26[2]	-14.896	-16.626	0.218	0.225	0.552	0.564
$E(X^{*\prime}\tilde{R})$[3]	1.383	1.358	1.033	1.025	1.007	1.002
Certainty Equivalent	1.209	1.145	1.022	0.998	1.001	0.996

[1] *Numbers are not comparable across levels of risk aversion, because for each level of risk aversion a different set of available assets was used: a set giving an exact value of opportunity cost typical for that level of risk aversion.*
[2] *The 26th asset is risk-free in nominal terms.*
[3] *Monthly gross expected returns on portfolios.*

Table 4

Illustrative optimal portfolio shares for unconstrained and constrained portfolio strategies for different values of relative risk aversion for nine assets[1]

# of An Asset	Relative Risk Aversion, $(1-\gamma)$, equal to 0.7		Relative Risk Aversion, $(1-\gamma)$, equal to 11		Relative Risk Aversion, $(1-\gamma)$, equal to 31	
	Unconstrained	Constrained	Unconstrained	Constrained	Unconstrained	Constrained
1	0.270	0.229	0.292	0.253	0.027	0.021
2	0.004	0.262	0.100	0.074	-0.051	-0.056
3	-0.199	0.037	0.590	0.514	0.038	0.027
4	0.598	0.252	0.110	0.127	0.022	0.047
5	0.001	0.015	0.162	0.082	-0.006	-0.003
6	0.472	0.786	-0.536	-0.450	0.040	0.037
7	0.231	0.202	-0.306	-0.329	0.013	0.003
8	5.234	3.659	0.280	0.242	0.055	0.047
9[2]	-5.612	-4.442	0.307	0.487	0.862	0.876
$E(X^{*\prime}\tilde{R})$[3]	1.079	1.058	1.027	1.018	1.004	1.001
Certainty Equivalent	1.056	1.020	1.016	1.009	1.002	0.998

[1] *Numbers are not comparable across levels of risk aversion, because for each level of risk aversion a different set of available assets was used: a set giving an exact value of opportunity cost typical for that level of risk aversion.*
[2] *The 9th asset is risk-free in nominal terms.*
[3] *Monthly gross expected returns on portfolios.*

Also Table 3 and Table 4 show expected returns on unconstrained and constrained optimal portfolios, $E\ (X^{*\prime}\tilde{R})$, for the three levels of relative risk aversion (0.7, 11 and 31).

The expected returns for constrained and unconstrained optimal portfolios for risk aversion of 0.7 are very large for 26-asset portfolios and somewhat large for nine-asset portfolios (comparing to initial wealth set equal to *1*). Expected returns are of medium size for risk aversion of 11 and of small size for risk aversion of 31. Big magnitudes of expected portfolio returns for high risk-tolerance investors confirm the previously made conclusion about very aggressive short sale strategies. With initial wealth set equal to *1* these magnitudes suggest very leveraged portfolios (unconstrained as well as constrained). For investors with risk aversion of 11 and 31 there is, definitely, some short selling is going on too, but not as aggressive as for investors with risk aversion of 0.7. The less aggressive short selling for medium or high risk aversion leads to lower mean return portfolios. Simaan's expected portfolio returns are much lower that the ones I have predicted for 26-asset portfolios, but only a little bit lower than those for nine-asset portfolios.

The big difference between the expected portfolio returns with 26 assets and those with nine assets is due to the fact that the more assets are available for investors the more opportunities they have to seek higher mean while simultaneously increasing diversification.

In comparing unconstrained expected portfolio returns and constrained expected portfolio returns for the three levels of risk aversion for the two tables I find that unconstrained and constrained expected portfolio returns for risk aversion of 31 are very close to each other; for risk aversion of 11 they are somewhat close, but not very; for risk aversion of 0.7 unconstrained and constrained expected portfolio returns are not close at all. These unconstrained and constrained expected portfolio returns show that as risk aversion increases, the closer to each other expected returns on unconstrained and constrained portfolios are, and thus the more nearly indifferent an investor is between the unconstrained and constrained portfolio strategies.

Also Table 3 and Table 4 report the *certainty equivalents* calculated for the same three levels of relative risk aversion (0.7, 11 and 31). The *certainty equivalent*, (*CE*), is defined by

(16) $$\frac{1}{\gamma} CE^{\gamma} = \frac{1}{\gamma} w_0^{\gamma} E\left(\tilde{R}^{\gamma}\right)$$

and so, with $w_0=1$,

(17) $$CE = \left(E\left[\tilde{R}^{\gamma}\right]\right)^{\frac{1}{\gamma}}.$$

The certainty equivalent represents the amount of certain wealth that would be viewed with indifference to the optimal portfolio. It is computed for investors of different levels of risk aversion: low (of 0.7), medium (of 11) and high (of 31). The two tables show that as risk aversion increases the value of certainty equivalent decreases (for the unconstrained portfolio strategy as well as for the constrained). This suggests that as investors become more afraid of risk they use less risky portfolio strategies and will be expecting lower returns from those portfolios and, so, the certain amount of wealth they will be willing to except with indifference will decrease.

Table 3 and Table 4 show that magnitudes of portfolio shares for different levels of relative risk aversion, as well as for unconstrained and constrained portfolio strategies, are very different. Comparison of optimal portfolio shares across different levels of risk aversion is meaningless, since portfolio shares for different levels of relative risk aversion were calculated by using different sets of available assets for each level of risk aversion. But comparison of unconstrained and constrained optimal portfolios is very interesting (how similar are constrained and unconstrained asset holdings to each other?) and possible (constrained and unconstrained portfolio shares for a particular level of relative risk aversion correspond to the same set of available assets).

In terms of comparing unconstrained and constrained optimal portfolios, I find in Table 3 and Table 4 that absolute values of shares of unconstrained portfolios are, in almost all cases, bigger than those of constrained portfolios.

Table 5 and Table 6 present correlation coefficients and geometric distances calculated between unconstrained and constrained portfolio share vectors for different levels of relative risk aversion: low, (from 0.7 to three), medium (from nine to 12) and high (from 29 to 31), for all assets in ones' portfolio (26 and nine) and separately for the group of nominally risky assets only (asset #1 through asset #25 in Table 5 and asset #1 through asset #8 in Table 6).

Both tables show high correlation between unconstrained and constrained optimal portfolio shares for all levels of risk aversion. But correlation coefficients calculated for all assets in portfolios (26 or nine) are higher that those calculated for the nominally risky assets only.

This difference in correlation coefficients can be explained by the presence of Treasury bills in optimal portfolios of 26 and nine assets. For the low levels of risk aversion when investors use highly leveraged portfolios they go very short on Treasury bills (in constrained as well as unconstrained strategies), which results in very similar negative portfolio shares for the asset. This similarity has a strong effect on the correlation calculated over all assets shares including the Treasury bills share.

The geometric distance calculated for different values of risk aversion is another way to compare unconstrained and constrained portfolios. The greater the geometric distance between the two portfolios, in other words the further the unconstrained optimal portfolio is from the constrained optimal portfolio, the greater an investor's welfare loss

Table 5

Illustrative correlation coefficients and geometric distances for unconstrained and constrained portfolios for different levels of relative risk aversion for 26 assets[1]

Relative Risk Aversion, (1 - γ)	Correlation Coefficients for all 26 assets	Geometric Distance for 26 assets	Correlation Coefficients for the group of nominally risky Assets only (first 25 assets)	Geometric Distance for nominally risky assets only (first 25 assets)
0.7	0.967	7.382	0.907	7.374
0.999	0.923	5.240	0.912	4.945
2	0.968	2.068	0.869	1.945
3	0.958	2.046	0.964	1.906
9	0.976	0.459	0.969	0.438
10	0.926	0.301	0.921	0.287
11	0.946	0.215	0.941	0.209
12	0.962	0.169	0.956	0.140
29	0.978	0.139	0.947	0.134
30	0.983	0.084	0.965	0.076
31	0.993	0.082	0.989	0.081

[1] Calculated for different fixed sets of assets for different levels of risk aversion; in each case the asset set is the one giving an opportunity cost typical for that level of risk aversion.

Table 6

Illustrative correlation coefficients and geometric distances for unconstrained and constrained portfolios for different levels of relative risk aversion for nine assets[1]

Relative Risk Aversion, (1 - γ)	Correlation Coefficients for all 9 assets	Geometric Distance for 9 assets	Correlation Coefficients for the group of nominally risky assets only (first 8 assets)	Geometric Distance for nominally risky assets only (first 8 assets)
0.7	0.996	5.835	0.978	2.788
0.999	0.993	2.048	0.987	1.682
2	0.985	0.619	0.978	0.409
3	0.813	0.409	0.756	0.408
9	0.997	0.086	0.986	0.047
10	0.998	0.047	0.972	0.032
11	0.969	0.037	0.954	0.035
12	0.996	0.040	0.991	0.040
29	0.956	0.026	0.950	0.020
30	0.890	0.021	0.879	0.018
31	0.989	0.017	0.976	0.016

[1] Calculated for different fixed sets of assets for different levels of risk aversion; in each case the asset set is the one giving an opportunity cost typical for that level of risk aversion.

is likely to be if he must choose the constrained portfolio, and so the higher the opportunity cost for the investor is. Table 5 and Table 6 show that as risk aversion increases, as investors become less risk tolerant, the geometric distance between unconstrained and constrained portfolios decreases making unconstrained and constrained portfolios more similar to each other. Simaan reached the same conclusion for the case with the riskless asset though his geometric distances are greater in magnitude than mine for my nine-asset portfolios and somewhat close to mine for my 26-asset portfolios. So, if the unconstrained optimal portfolio and the constrained optimal portfolio are getting closer as risk aversion increases, it must be that the opportunity cost will decrease. That is exactly my finding from Table 1 and Table 2.

3.1.3. Regret in the worst-case scenario

Large negative and positive asset holdings (Table 3 and Table 4) in portfolios for investors with a level of risk aversion of 0.7 suggest that the investors take on a lot of risk. This raises the question: if the worst possible portfolio outcome occurs, then how much will the investors suffer from such an outcome? It is possible to measure the investors' proportionate regret from the worst-case scenario with such a risky portfolio.

Table 7 and Table 8 report the proportionate regret, $(\theta - 1)$, for 26 assets and for nine assets, that will be incurred by investors if the worst possible outcome of asset returns occurs. This θ is defined by

(18) $$U[\,\theta\,(X^{*\prime} R)^{\,worst}] = EU(X^{*\prime}\tilde{R})$$

where X^* is the optimally chosen portfolio, $(X^{*\prime}R)^{worst}$ is the one of the 120 states of nature giving the lowest portfolio return, $U(X^{*\prime}R^{worst})$ is an investor's utility from getting the worst possible portfolio outcome, $EU(X^{*\prime}\tilde{R})$ is an investor's ex ante expected utility.

For unconstrained investors (for the case with 26 assets as well as for nine assets) the mean of the proportionate regret (over 1,000 replications) is the highest for the low level of risk aversion of 0.7 and the lowest for the high level of risk aversion of 31. This means that high risk-tolerance investors do choose risky unconstrained asset allocations. And it is getting riskier as the number of assets increases. Those asset allocations are so risky at the level of risk aversion of 0.7, that if the worst possible outcome occurs it would require for investors with 26 assets to receive 1022.1% of initial wealth in compensation and for investors with nine assets to receive 525.7% of initial wealth in order to get the same level of ex post utility as their ex ante expected utility. For the high level of 31 for risk aversion the mean of the proportionate regret (over 1,000 replications) is 4.1% (0.041) for investors with 26 assets and 2.7% (0.027) for investors with nine assets. Such a low proportionate regret suggests that low risk-tolerance unconstrained investors choose very conservative unconstrained asset allocations. So conservative are their allocations that even the worst possible outcome will require for them less than 5.0% of initial wealth to get to the same level of utility as their ex ante expected utility.

For constrained portfolio strategies the mean proportionate regret (over 1,000 replications) ranges from 483.8% (4.838) for risk aversion of 0.7 to 3.7% (0.037) for risk aversion of 31 for investors with 26 assets, and from 348.9% (3.489) for risk aversion of 0.7 to 1.9% (0.019) for risk aversion of 31 for investors with nine assets. This means that constrained portfolios have a very restrictive character and do not let high risk-tolerance

Table 7

The ex post proportionate regret, (θ - *1*), under the worst portfolio outcome for 26 assets

Relative Risk Aversion, (1-γ)	Portfolios	Smallest	Mean	Median	Largest	Standard Deviation
0.7	*Unconstrained*	0.146	10.221	9.310	41.766	14.551
	Constrained	0.112	4.838	3.083	16.435	7.772
11	*Unconstrained*	0.058	0.135	0.120	0.243	0.024
	Constrained	0.049	0.113	0.111	0.155	0.019
31	*Unconstrained*	0.023	0.041	0.040	0.076	0.007
	Constrained	0.012	0.037	0.035	0.054	0.006

Table 8

The ex post proportionate regret, (θ - *1*), under the worst portfolio outcome for nine assets

Relative Risk Aversion, (1-γ)	Portfolios	Smallest	Mean	Median	Largest	Standard Deviation
0.7	*Unconstrained*	0.162	5.257	4.425	25.509	3.148
	Constrained	0.103	3.489	2.916	9.346	2.654
11	*Unconstrained*	0.011	0.069	0.061	0.240	0.018
	Constrained	0.009	0.048	0.039	0.153	0.017
31	*Unconstrained*	0.009	0.027	0.025	0.069	0.007
	Constrained	0.008	0.019	0.017	0.048	0.006

investors take a lot of risk. For low risk-tolerance investors constrained portfolios are somewhat close to unconstrained portfolios (the difference in the mean regret is less than 1%) and represent very conservative asset allocations with very little risk to take. Note that the tendency for the mean-variance efficiency constraint to make portfolios more conservative is also seen in Table 3 and Table 4, which show that at each level of risk aversion the mean portfolio return is less when the constraint is present than when it is not.

4. Conclusion

In this paper I have investigated the opportunity cost incurred by investors when they use constrained optimal mean-variance efficient portfolios instead of unconstrained optimal portfolios. The original historical returns were used. CRRA utility function and the proportionate opportunity cost have been used. The opportunity cost has been calculated for different values of relative risk aversion (including extreme levels of relative risk aversion) for 26-asset portfolios and nine-asset portfolios. The highest mean across simulations of the proportionate opportunity cost found is 5.6% (0.056) for the level of relative risk aversion of 0.7 for 26-asset portfolios. The lowest mean of the proportionate opportunity cost found is 0.4% (0.004) for the level of relative risk aversion of 31 for the nine-asset portfolio. As the level of relative risk aversion increases the proportionate opportunity cost decreases.

My findings are different from those of Simaan (1993) and Tew, Reid and Witt (1991) in several ways. First, they differ in the magnitude of the opportunity cost: the highest opportunity cost Simaan found is 30% for the case with no riskless asset and

0.5% for the case with the riskless asset; the highest mean opportunity cost I found (with the nominally riskless asset included) is 5.6% for the 26-asset portfolio and 3.5% for the nine-asset portfolio. Second, Simaan found that when the market offers a riskless asset, no matter what the level of risk aversion is the opportunity cost is almost zero. For the case with no riskless asset Simaan's and Tew, Reid and Witt's opportunity cost increases as risk aversion increases. I have worked with a semi-riskless asset: in nominal terms returns on Treasury bills are riskless, but in real terms there is no riskless asset (inflation is uncertain in any period and, thus, so are real returns on Treasury bills). So, with a semi-riskless asset I found that the opportunity cost decreases as risk aversion increases and it becomes almost zero for relative risk aversion greater than 29 for nine-asset portfolios. The differences in magnitude that I found between my results and Simaan' are due to several factors. I have used utility functions with constant relative risk aversion preferences, not with constant absolute risk aversion preferences. I also used a wide variety of levels of relative risk aversion (from 0.7 to 31), thus getting a range of values of the opportunity cost. Simaan on the other hand used levels of relative risk aversion of two and greater arguing that levels of risk aversion less than two offer very aggressive infeasible investment strategies.

Tew, Reid and Witt (1991) worked only with portfolios of risky assets (with the number of assets increasing from two to nine) and relative risk aversion that ranged from 0.1 to 1.9 (along with CRRA utility functions). They found very small opportunity costs for all considered levels of risk aversion. Also they found that as risk aversion increases the opportunity cost increases too. Their findings are consistent with Simaan's case where there was no riskless asset introduced. They also found that as the number of assets in portfolio increases the opportunity cost decreases. Tew, Reid and Witt's results are not consistent with mine, because even though I worked with 26-asset portfolios and nine-asset portfolios I also included a semi-riskless asset in both portfolios, which Tew, Reid and Witt did not do.

Therefore, based on my calculations, I may conclude that for investors with very high levels of relative risk aversion (29 and above) mean-variance analysis performs very well (with a relatively small number of assets in ones' portfolio) or significantly better (with a large number of assets in one's portfolio) than for investors with medium or low levels of relative risk aversion. So, as risk aversion increases mean-variance strategies show a fairly good approximation to the optimal portfolio strategy.

References:

Campbell, J. and L. Viceira, 2002. *Strategic Asset Allocation* (Oxford University Press)

Merton, R., 1972. An analytic Derivation of the Efficient Portfolio Frontier, *Journal of Financial and Quantitative Analysis*, 1851-1872.

Simaan, Y., 1993. What Is the Opportunity Cost of Mean-Variance Investment Strategy? *Management Science* 39, 578-587.

Tew, B., D. Reid, and C. Witt, 1991. The Opportunity Cost of a Mean-Variance Efficient Choice, *The Financial Review* 26, 31-43.